26. Characterizations of BCI, BCK-Algebras

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(Comm. by Kinjirô KUNUGI, M.J.A., Feb. 12, 1966)

In this note, we shall consider some characterizations of the *BCI*, *BCK*-algebras defined in [1]. By a *BCI-algebra*, we mean an algebra $M=\langle X, 0, *\rangle$ with an element 0 and a binary operation $*$ satisfying the following conditions *BCI* 1~5:

BCI 1　$(x*y)*(x*z)\leqslant z*y$,

BCI 2　$x*(x*y)\leqslant y$,

BCI 3　$x\leqslant x$,

BCI 4　$x\leqslant y, y\leqq x$ imply $x=y$.

BCI 5　$x\leqslant 0$ implies $x=0$,

where $x\leqslant y$ means $x*y=0$.

BCI 5 is equivalent to: $x*0=0$ implies $x=0$. If *BCI* 5 is replaced by *BCI* 6: $0\leqslant x$ for every $x\in X$, the algebra M is called *BCK-algebra*.

In [1], we proved that

(6)　$(y*x)*(z*x)\leqslant y*z$

holds in the *BCI*-algebra. We first prove the following

Theorem 1. *The BCI-algelra is characterized by BCI* 2~5 *and* (6).

Proof. (6) implies the following results:

(7)　If $y\leqslant z$, then $y*x\leqslant z*x$.

(8)　If $x\leqslant y$, $y\leqslant z$, then $x\leqslant z$.

By (6) and (7), we have

(9)　$((y*x)*(z*x))*u\leqslant(y*z)*u$.

We substitute $y*u$ for z in (9), then by *BCI* 2, we have

(10)　$(y*x)*((y*u)*x)\leqslant u$.

In formula (10), let $x=y$, $y=x*z$, and $u=(x*y)*z$, then

$$((x*z)*y)*(((x*z)*((x*y)*z))*y)\leqslant(x*y)*z.$$

The second term of the left side is equal to 0, hence if $x*y\leqslant z$, i.e. $(x*y)*z=0$, then in the formula above, the right side is equal to 0, so we have $x*z\leqslant y$ by *BCI* 5. Therefore, we have

(11)　If $x*y\leqslant z$, then $x*z\leqslant y$.

Hence, by (11) and (6),

BCI 1　$(x*y)*(x*z)\leqslant z*y$.

It is obvious from [1] that the converse holds. The proof of Theorem 1 is complete.

By Theorem 1, we have the following

Theorem 2. *A BCK-algebra is characterized by BCI* 2~4, *BCI* 6 *and* (6).

In [1], we proved that

(12) $(x*y)*(x*(z*(u*y)))\leqslant z*u$

holds in the *BCI*-algebra. We show the following

Theorem 3. *A BCI-algebra is characterized by BCI* 3~5 *and* (12).

Therefore we also have the following

Theorem 4. *A BCK-algebra is characterized by BCI* 3, 4, 6, *and* (12).

We shall prove Theorem 3.

Proof. In (12), let $x=x*(x*y)$, $z=u=x$, then

$$((x*(x*y))*y)*((x*(x*y))*(x*(x*y)))\leqslant x*x.$$

Then by *BCI* 3 and 5, we have $x*(x*y)\leqslant y$, which is *BCI* 2. From (12), we have the following result

(13) if $x\leqslant z*(u*y)$ and $z\leqslant u$, then $x\leqslant y$.

In (12), if $z=u$, then $x*y\leqslant x*(z*(z*y))$. This formula has the form of $x\leqslant z*(u*y)$ by putting $x*y=x$, $x=z$, $z=u$, and $u*y=y$. Hence if $x\leqslant z$ (this means $z\leqslant u$ in (13)), then $x*y\leqslant z*y$. Therefore we have

(7) If $x\leqslant y$, then $x*z\leqslant y*z$.

(8) If $x\leqslant y$, $y\leqslant z$, then $x\leqslant z$.

Let $u=z$ in (12), then $x*y\leqslant x*(z*(z*y))$, and by (7), we have

(14) $(x*y)*u\leqslant (x*(z*(z*y)))*u$.

Next, put $x=y*x$, $y=z*x$, $z=y$, and $u=y*z$ in (14), then

$$((y*x)*(z*x))*(y*z)\leqslant ((y*x)*(y*(y*(z*x))))*(y*z).$$

The right side of the above formula is equal to 0, since it is obtained by substituting y for x and z, x for y and z for u in (12). Hence we have

(6) $(y*x)*(z*x)\leqslant y*z$.

Therefore by Theorem 1, we have Theorem 3 and complete the proof.

Remark. We shall show that *BCI* 1~5, (6), (8), *and* (11) *imply* $(x*y)*z\leqslant (x*z)*y$, *and hence* $(x*y)*z=(x*z)*y$.

By *BCI* 1, (6) and (8), we have $((x*y)*z)*((x*u)*z)\leqslant (x*y)*(x*u)\leqslant u*y$. Hence by (11),

$$((x*y)*z)*(u*y)\leqslant (x*u)*z.$$

Let $u=x*z$ in the formula above, then

$$((x*y)*z)*((x*z)*y)\leqslant (x*(x*z))*z.$$

Then the right side is equal to 0, hence we have

$$(x*y)*z\leqslant (x*z)*y.$$

By *BCI* 4, we have $(x*y)*z=(x*z)*y$. A proof of this formula is given in [1].

Reference

[1] K. Iséki: An algebra related with a propositional calculus. Proc. Japan Acad., **42**, 26—29 (1966).