24. *Axiom Systems of B-algebra. IV*

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In our previous notes (see [1], [2], [3], [4], and [5]), we considered how to formulate axiom systems of propositional calculi into algebraic forms. Among these algebras, we are concerned with the B-algebra which is equivalent to the notion of Boolean algebra. The purpose of our paper is to give some axiom systems of the B-algebra following our new point of view.

In his note (see [4]), K. Iséki defined the B-algebra. Let $\boldsymbol{M}=\langle X, 0, *, \sim\rangle$ be a B-algebra, i.e., $\boldsymbol{M}$ is an abstract algebra which satisfies the following axioms:

$B\,1$　$x*y\leqslant x$,

$B\,2$　$(x*z)*(y*z)\leqslant(x*y)*z$,

$B\,3$　$x*y\leqslant(\sim y)*(\sim x)$,

$B\,4$　$0\leqslant x$,

where $x\leqslant y$ means $x*y=0$, and if $x\leqslant y$, $y\leqslant x$, then we write $x=y$. As already shown in [1] and [3], the above axiom system is equivalent to the following axioms:

$F\,1$　$x*y\leqslant x$,

$F\,2$　$(x*y)*(z*y)\leqslant(x*z)*y$,

$F\,3$　$(\sim x)*(\sim y)\leqslant y*x$,

$F\,4$　$x\leqslant\sim(\sim x)$,

$F\,5$　$\sim(\sim x)\leqslant x$,

$D\,1$　$0\leqslant x$,

$D\,2$　If $x\leqslant y$ and $y\leqslant x$, then we put $x=y$,

$D\,3$　$x\leqslant y$ means $x*y=0$.

The conditions $F\,1\sim F\,5$ are an algebraic formulation of Frege axioms of classical propositional calculus (see [6]). Therefore a B-algebra is characterized by $F\,1\sim F\,5$ and $D\,1$, $D\,2$, $D\,3$.

In this note, we shall show the following

Theorem. *A B-algebra* $\boldsymbol{M}=\langle X, 0, *, \sim\rangle$ *is characterized by*

$L\,1$　$(x*y)*(x*z)\leqslant z*y$,

$L\,2$　$x\leqslant x*(\sim x)$,

$L\,3$　$x*(\sim y)\leqslant y$,

and $D\,1$, $D\,2$, $D\,3$.

The conditions $L\,1\sim L\,3$ are an algebraic formulation of the two valued classical propositional calculus given by J. Lukasiewicz (see [6]).

First a proof of F$\Rightarrow$L will be given by using the technique in the note of Y. Arai and K. Iséki (see [1]). After giving a proof of F$\Rightarrow$L, we shall prove L$\Rightarrow$F. We shall give a proof of F$\Rightarrow$L.

By $F2$, $F4$, $F5$, $D1$, $D2$, and $D3$, we have the following lemmas:

Lemma 1. $x=\sim(\sim x)$,

Lemma 2. $x*z\leqslant y$ *implies* $x*y\leqslant z*y$.

Put $x=((x*y)*(z*y))*((x*z)*y)$ and $y=x*z$ in $F1$, then we have

$$(((x*y)*(z*y))*((x*z)*y))*(x*z)\leqslant((x*y)*(z*y))*((x*z)*y).$$

The right side is equal to 0 by $F2$ and $D3$. Hence, by $D1$ and $D3$, we have

(1) $((x*y)*(z*y))*((x*z)*y)\leqslant x*z$.

By (1) and Lemma 2, we have

$$((x*y)*(z*y))*(x*z)\leqslant((x*z)*y)*(x*z).$$

For the right side is equal to 0 by substituting $x*z$ for x in $F1$ and $D3$, from $D1$ and $D3$, we have

(2) $(x*y)*(z*y)\leqslant x*z$.

(2) implies

Lemma 3. $x\leqslant z$ *implies* $x*y\leqslant z*y$.

Lemma 4. $x\leqslant z$ *and* $z\leqslant y$ *imply* $x\leqslant y$.

Put $x=\sim x, y=\sim y$ in $F3$ and use Lemma 1, then we have

(3) $x*y\leqslant(\sim y)*(\sim x)$.

By substituting $\sim y$ for x and $\sim x$ for y in $F1$, we have $(\sim y)*(\sim x)\leqslant\sim y$. Applying Lemma 4, we have

(4) $x*y\leqslant\sim y$.

Put $y=\sim y$ in (4), then we have $x*(\sim y)\leqslant\sim(\sim y)$. Applying Lemma 1 to the right side, we have

(5) $x*(\sim y)\leqslant y$.

Put $x=(x*y)*x$ and $y=(z*y)*(x*y)$ in $F1$, then we have

$$((x*y)*x)*((z*y)*(x*y))\leqslant(x*y)*x.$$

Since the right side is equal to 0 by $F1$ and $D3$, form $D1$ and $D3$, we have

(6) $(x*y)*x\leqslant(z*y)*(x*y)$.

Put $x=x*y, y=z$, and $z=z*y$ in (2) and use Lemma 2, then we have

$$((x*y)*z)*((x*y)*(z*y))\leqslant((z*y)*z)*((x*y)*(z*y)).$$

The right side is equal to 0 by putting $x=z$ and $z=x$ in (6) and $D3$. Therefore, by $D1$ and $D3$, we have

(7) $(x*y)*z\leqslant(x*y)*(z*y)$.

Applying Lemma 3 to (7), we have

$$((x*y)*z)*((x*z)*y)\leqslant((x*y)*(z*y))*((x*z)*y).$$

The right side is equal to 0 by $F2$ and $D3$. Then, by $D1$ and $D3$, we have

(8) $(x*y)*z \leqslant (x*z)*y$.

Then we have Lemma 5 which is called the *commutative law*:

Lemma 5. *$x*z \leqslant y$ implies $x*y \leqslant z$.*

By (2) and the commutative law, we have

(9) $(x*y)*(x*z) \leqslant z*y$.

Applying Lemma 3 to (9), we have

$$((x*y)*(x*z))*((\sim y)*(\sim z)) \leqslant (z*y)*((\sim y)*(\sim z)).$$

The right side is equal to 0 by substituting z for x in (3) and $D\,3$. Hence, by $D\,1$ and $D\,3$, we have

(10) $(x*y)*(x*z) \leqslant (\sim y)*(\sim z)$.

Applying Lemma 2 to (4), we have

(11) $x*(\sim y) \leqslant y*(\sim y)$.

Applying Lemma 3 to (10), we have

$$((x*y)*(x*z))*(z*(\sim z)) \leqslant ((\sim y)*(\sim z))*(z*(\sim z)).$$

The right side is equal to 0 by substituting $\sim y$ for x and z for y in (11) and $D\,3$. Therefore, by $D\,1$ and $D\,3$, we have

(12) $(x*y)*(x*z) \leqslant z*(\sim z)$.

Applying Lemma 5 to $F\,1$, we have

(13) $x*x \leqslant y$, i.e. $x \leqslant x$.

By putting $y=x*(\sim x)$ and $z=x$ in (12) and applying Lemma 5, we have $(x*(x*(\sim x)))*(x*(\sim x)) \leqslant x*x$. The right side is equal to 0 by (13) and $D\,3$. Then, by $D\,1$ and $D\,3$, we have

(14) $x*(x*(\sim x)) \leqslant x*(\sim x)$.

Put $x=x*x$ and $y=(y*x)*x$ in $F\,1$, then we have

$$(x*x)*((y*x)*x) \leqslant x*x.$$

The right side is equal to 0 by (13) and $D\,3$. Then, by $D\,1$ and $D\,3$, we have

(15) $x*x \leqslant (y*x)*x$.

Applying Lemma 2 to $F\,2$, we have

(16) $(x*y)*((x*z)*y) \leqslant (z*y)*((x*z)*y)$.

Put $x=y$, $y=x$, and $z=x$ in (16), then we have

$$(y*x)*((y*x)*x) \leqslant (x*x)*((y*x)*x).$$

The right side is equal to 0 by (15) and $D\,3$. Then, by $D\,1$ and $D\,3$, we have

(17) $y*x \leqslant (y*x)*x$.

Put $x=x*(\sim x)$ and $y=x$ in (17), then we have

$$x*(x*(\sim x)) \leqslant (x*(x*(\sim x)))*(x*(\sim x)).$$

The right side is equal to 0 by (14) and $D\,3$. Therefore, by $D\,1$ and $D\,3$, we have

(18) $x \leqslant x*(\sim x)$.

We have proved that $F\,1 \sim F\,5$ imply $L\,1 \sim L\,3$, i.e. (5), (9), and (18). Next we shall prove that $F\,1 \sim F\,5$ is derived from $L\,1 \sim L\,3$.

By $L\,1$, $D\,1$, and $D\,3$, we have

Lemma 1′. *$z\leqslant y$ implies $x*y\leqslant x*z$,*

Lemma 2′. *$x\leqslant z, z\leqslant y$ imply $x\leqslant y$.*

From $L\,2$, $L\,3$, and $D\,2$, we have

Lemma 3′. $x*(\sim x)=x$.

In $L\,1$, put $y=x$ and $z=x*(\sim x)$, then we have

$$(x*x)*(x*(x*(\sim x)))\leqslant(x*(\sim x))*x.$$

By substituting x for y in $L\,3$, the right side is equal to 0, and by $L\,2$ and $D\,3$, the second term of the left side is equal to 0. Hence, from $D\,1$ and $D\,3$, we have

Lemma 4′. $x*x=0$, i.e. $x\leqslant x$.

Applying Lemma 1′ to $L\,3$, we have

(1′) $x*y\leqslant x*(x*(\sim y))$.

If we put $y=\sim x$ and $z=\sim y$ in $L\,1$, we have

$$(x*(\sim x))*(x*(\sim y))\leqslant(\sim y)*(\sim x).$$

Next we substitute $\sim y$ for x and x for y in $L\,3$, then we have $(\sim y)*(\sim x)\leqslant x$. Hence, applying Lemma 2′, we have

$$(x*(\sim x))*(x*(\sim y))\leqslant x.$$

Therefore, from Lemma 3′, we get

(2′) $x*(x*(\sim y))\leqslant x$.

(2′) and Lemma 1′ imply $(x*y)*x\leqslant(x*y)*(x*(x*(\sim y)))$. By (1′) and $D\,3$, the right side is equal to 0. Hence, by $D\,1$ and $D\,3$, we have

(3′) $x*y\leqslant x$.

Put $x=y, y=z$ in (3′) and use Lemma 1′, then we have

(4′) $x*y\leqslant x*(y*z)$.

By (4′), $D\,1$ and $D\,3$, we have

Lemma 5′. *$x\leqslant y*z$ implies $x\leqslant y$.*

If we substitute $\sim x$ for y and y for z in $L\,1$, we have

$$(x*(\sim x))*(x*y)\leqslant y*(\sim x).$$

Using Lemma 3′, then we have

(5′) $x*(x*y)\leqslant y*(\sim x)$.

(5′) and Lemma 5′ imply

(6′) $x*(x*y)\leqslant y$.

We shall now prove the commutative law, i.e. $x*z\leqslant y$ implies $x*y\leqslant z$. Let $(x*z)*y=0$, i.e. $x*z\leqslant y$, then we have $x*y\leqslant x*(x*z)$ from Lemma 1′. Applying Lemma 2′, we have $x*y\leqslant z$. Hence we have

Lemma 6′. *$x*z\leqslant y$ implies $x*y\leqslant z$.*

By $L\,1$ and Lemma 6′, we have

(7′) $(x*y)*(z*y)\leqslant x*z$.

By $L\,3$ and Lemma 6′, we have

(8′) $x*y\leqslant\sim y$.

By (8′) and Lemma 1′, we have $x*(\sim y)\leqslant x*(x*y)$. Next, by the above formula, (5′) and Lemma 2′, we have

(9′) $x*(\sim y)\leqslant y*(\sim x)$.

Let $x=y$ and $y=x$ in (9′), then $y*(\sim x)\leqslant x*(\sim y)$. Therefore, by considering $D\,2$, we have

Lemma 7′. $y*(\sim x)=x*(\sim y)$.

By (5′) and Lemma 7′, we have

(10′) $x*(x*y)\leqslant x*(\sim y)$.

Put $x=\sim x$ and $y=x$ in (10′), then we have

$$(\sim x)*((\sim x)*x)\leqslant(\sim x)*(\sim x).$$

For the right side is equal to 0 by Lemma 4′, by $D\,1$ and $D\,3$, we have

(11′) $\sim x\leqslant\sim x*x$.

Put $x=\sim(\sim x)$, $y=x$ in $L\,3$ and use Lemma 1′, then we have

$$\sim(\sim x)*x\leqslant\sim(\sim x)*(\sim(\sim x)*(\sim x)).$$

The right side is equal to 0 by substituting $\sim x$ for x in (11′) and $D\,3$. Hence, by $D\,1$ and $D\,3$, we have

(12′) $\sim(\sim x)\leqslant x$.

Put $y=\sim x$ in (8′) and use Lemma 1′, we have $x*(\sim(\sim x))\leqslant x*(x*(\sim x))$. The right side is equal to 0 by $L\,2$ and $D\,3$. Hence, by $D\,1$ and $D\,3$, we have

(13′) $x\leqslant\sim(\sim x)$.

(12′), (13′), and $D\,2$ show

Lemma 8′. $x=\sim(\sim x)$.

Put $x=\sim x$ in (9′), then we have $(\sim x)*(\sim y)\leqslant y*(\sim(\sim x))$. The second term of the right side is equal to x by Lemma 8′. Hence we have

(14′) $(\sim x)*(\sim y)\leqslant y*x$.

Put $x=x*y$, $y=u$, $z=z*y$ in (7′), use (7′) and apply Lemma 2′, then we have $((x*y)*u)*((z*y)*u)\leqslant x*z$. Hence, applying the commutative law, i.e. Lemma 6′, we have

(15′) $((x*y)*u)*(x*z)\leqslant(z*y)*u$.

(15′) means

Lemma 9′. *$z*y\leqslant u$ implies $(x*y)*u\leqslant x*z$.*

Put $x=x*y$ in (10′), then we have

$$(x*y)*((x*y)*y)\leqslant(x*y)*(\sim y).$$

The right side is equal to 0 by (8′) and $D\,3$. Then, by $D\,1$ and $D\,3$, we have

(16′) $x*y\leqslant(x*y)*y$.

Put $x=x*y$, $z=x*z$ in (7′), use $L\,1$ and apply Lemma 2′, then we have

$$((x*y)*y)*((x*z)*y)\leqslant z*y.$$

Using the commutative law, i.e. Lemma 6′, we have

(17′) $((x*y)*y)*(z*y)\leqslant(x*z)*y$.

By (17′) and Lemma 9′, we have

$$((x*y)*(z*y))*((x*z)*y)\leqslant(x*y)*((x*y)*y).$$

The right side is equal to 0 by (16′) and $D3$. Therefore, by $D1$ and $D3$, we have

(18′) $(x*y)*(z*y)\leqslant(x*z)*y$.

We have proved that $L1\sim L3$ imply $F1\sim F5$, i.e. (3′), (12′), (13′), (14′), and (18′). Now we have completed the proof of F$\Longleftrightarrow$L.

References

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