52. On Axiom Systems of Propositional Calculi. XVI

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In this paper, we shall show that several axiom systems of positive implicational calculus are equivalent. First we shall prove that 2-Axiom Base given by J. Lukasiewicz implies other axiom basis, i.e. 4-Axiom Base and 3-Axiom Base given by D. Hilbert, 2-Axiom Base and some 1-Axiom Basis given by C. A. Meredith (for example see, [2]).

For the details of the notations and the two rules of inferences for deductions, see [1].

The 2-Axiom Base given by J. Lukasiewicz is the set of the following two formulas.

1 $CpCqp$.

2 $CCpCqrCCpqCpr$.

Under these axioms, we have:

1 $p/CCpCqrCCpqCpr,\ q/Cqr\ {*}C2$—3,

3 $CCqrCCpCqrCCpqCpr$.

2 $p/Cqr,\ q/CpCqr,\ r/CCpqCpr\ {*}C3$—$C1\ p/Cqr,\ q/p$—4,

4 $CCqrCCpqCpr$.

2 $p/Cqr,\ q/Cpq,\ r/Cpr\ {*}C4$—5,

5 $CCCqrCpqCCqrCpr$.

1 $p/CCCqrCpqCCqrCpr,\ q/Cpq\ {*}C5$—6,

6 $CCpqCCCqrCpqCCqrCpr$.

2 $p/Cpq,\ q/CCqrCpq,\ r/CCqrCpr\ {*}C6$—$C1\ p/Cpq,$
q/Cqr—7,

7 $CCpqCCqrCpr$.

1 $p/CqCpq,\ q/CCpqCpr\ {*}C3\ p/q,\ q/p$—8,

8 $CCCpqCprCqCpq$.

2 $p/CCpqCpr,\ q/CqCpq,\ r/CqCpr\ {*}C4\ p/q,\ q/Cpq,$
r/Cpr—$C8$—9,

9 $CCCpqCprCqCpr$.

4 $p/CpCqr,\ q/CCpqCpr,\ r/CqCpr\ {*}C9$—$C2$—10,

10 $CCpCqrCqCpr$.

2 $p/CpCqr,\ q/Cpq,\ r/Cpr\ {*}C2$—11,

11 $CCCpCqrCpqCCpCqrCpr$.

2 $r/p\ {*}C1$—12,

12 $CCpqCpp$.

12 $q/Cqp\ {*}C1$—13,

13 $Cpp.$

1 $p/Cpp,\ q/CpCqp\ {}^*C13—14,$

14 $CCpCqpCpp.$

11 $q/p,\ r/q\ {}^*C14—15,$

15 $CCpCpqCpq.$

The set of theses 1, 4, 10, and 15 is the 4-Axiom Base given by D. Hilbert. The 3-Axiom Base given by D. Hilbert consists of theses 1, 7, and 15.

10 $p/CpCqr,\ q/Cpq,\ r/Cpr\ {}^*C2—16,$

16 $CCpqCCpCqrCpr.$

The set of theses 1, 16 is the 2-Axiom Base given by C. A. Meredith

7 $p/q,\ q/Cpq\ {}^*C3\ p/q,\ q/p—17,$

17 $CCCpqrCqr.$

7 $p/Cqr,\ q/CCqCrtCqt,\ r/CsCCqCrtCqt\ {}^*C16\ p/q,\ q/r,$
$r/t—C1\ p/CCqCrtCqt—18,$

18 $CCqrCsCCqCrtCqt.$

7 $p/CCpqr,\ q/Cqr,\ r/CsCCqCrtCqt\ {}^*C17—C18—19,$

19 $CCCpqrCsCCqCrtCqt.$

This thesis is the 1-Axiom Base given by C. A. Meredith.

7 $q/Csp,\ r/Cqr\ {}^*C1\ q/s—20,$

20 $CCCspCqrCpCqr.$

7 $p/CCspCqr,\ q/CpCqr,\ r/Cpr\ {}^*C20—21,$

21 $CCCpCqrCprCCCspCqrCpr.$

7 $p/Cpq,\ q/CCpCqrCpr,\ r/CCCspCqrCpr\ {}^*C16—$
$C21—22,$

22 $CCpqCCCspCqrCpr.$

1 $p/CCpqCCCspCqrCpr,\ q/t\ {}^*C22—23,$

23 $CtCCpqCCCspCqrCpr.$

This thesis is also a 1-Axiom Base given by C. A. Meredith.

Next we shall show that 4-Axiom Base given by D. Hilbert implies the 2-Axiom Base by J. Lukasiewicz. The set of the following four theses is the 4-Axiom Base.

1 $CCpCpqCpq,$

2 $CCqrCCpqCpr,$

3 $CCpCqrCqCpr,$

4 $CpCqp.$

3 $p/Cqr,\ q/Cpq,\ r/Cpr\ {}^*C2—5,$

5 $CCpqCCqrCpr.$

2 $p/s,\ q/Cpq,\ r/CCqrCpr\ {}^*C5—6,$

6 $CCsCpqCsCCqrCpr.$

5 $p/CsCpq,\ q/CsCCqrCpr,\ r/CCqrCsCpr\ {}^*C6—C3$
$p/s,\ q/Cqr,\ r/Cpr—7,$

7 *CCsCpqCCqrCsCpr.*

5 *p/CpCqr, q/CqCpr, r/CCpqCpCpr* **C*3—*C*2 *r/Cpr*—8,

8 *CCpCqrCCpqCpCpr.*

7 *s/CpCqr, p/Cpq, q/CpCpr, r/Cpr* **C*8—*C*1 *q/r*—9,

9 *CCpCqrCCpqCpr.*

The set of theses 4 and 9 is the 2-Axiom Base by J. Lukasiewicz. Hence the proof is complete.

Further we shall prove that the 3-Axiom Base by D. Hilbert implies the 4-Axiom Base by him. The 3-Axiom Base consists of the following three formulas.

1 *CCpCpqCpq,*

2 *CCpqCCqrCpr,*

3 *CpCqp.*

2 *p/CCpqp, q/CCpqCCCpqq, r/CCpqq* **C*2 *p/Cpq, q/p, r/q*—*C*1 *p/Cpq*—4,

4 *CCCpqpCCpqq.*

2 *q/CCpqp, r/CCpqq* **C*3 *q/Cpq*—*C*4—5,

5 *CpCCpqq.*

2 *q/CCpqq* **C*5—6,

6 *CCCCpqqrCpr.*

2 *p/CpCqr, q/CCCqrrCpr, r/CqCpr* **C*2 *q/Cqr*—*C*6 *p/q, q/r, r/Cpr*—7,

7 *CCpCqrCqCpr.*

7 *p/Cpq, q/Cqr, r/Cpr* **C*2—8,

8 *CCqrCCpqCpr.*

Theses 1, 3, 7, and 8 are the 4-Axiom Base by D. Hilbert.

Finally we shall prove that the A-xiom Base by C. A. Meredith implies the 2-Axiom Base by J. Lukasiewicz. The 2-Axiom Base by C. A. Meredith is given as the followings:

1 *CpCqp,*

2 *CCpqCCpCqrCpr.*

2 *p/q, q/Cpq* **C*1 *p/q, q/p*—3,

3 *CCqCCpqrCqr.*

1 *p/CCpqCCpCqrCpr* **C*2—4,

4 *CqCCpqCCpCqrCpr.*

3 *r/CCpCqrCpr* **C*4—5,

5 *CqCCpCqrCpr.*

5 *p/s, q/CqCCpCqrCpr, r/t* **C*5—6,

6 *CCsCCqCCpCqrCprtCst.*

6 *s/CpCqr, t/CqCpr* **C*5 *q/CpCqr, p/q, r/Cpr*—7,

7 *CCpCqrCqCpr.*

7 *p/Cpq, q/CpCqr, r/Cpr* **C*2—8,

8 *CCpCqrCCpqCpr*.

The set of theses 1 and 8 is the 2-Axiom Base by J. Lukasiewicz. Therefore the proof is complete.

References

[1] Y. Imai and K. Iséki: On axiom systems of propositional calculi. I. Proc. Japan Acad., **41**, 436-439 (1965).

[2] C. A. Meredith and A. N. Prior: Notes on the axiomatics of the propositional calculus. Notre Dame Journal of Formal Logic, **4**, 171-187 (1963).