51. On Axiom Systems of Propositional Calculi. XV

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In his article on the protothetic [1], S. Leśniewski considered a new calculus called the equivalential calculus. This calculus is formulated as follows: Let M be an abstract set with the only undefined truth functor $\equiv$ as a primitive notion. If the set $\boldsymbol{M}=\langle M, \equiv\rangle$ satisfies the following conditions:

1 $p\equiv r.\equiv.q\equiv p: r\equiv q,$

2 $p\equiv.q\equiv r:\ \equiv: p\equiv q.\equiv r,$

then $\boldsymbol{M}$ is called the *equivalential calculus*.

By using the bracket, the conditions above are written in the form of

1 $((p\equiv r)\equiv(q\equiv p))\equiv(r\equiv q),$

2 $(p\equiv(q\equiv r))\equiv((p\equiv q)\equiv r).$

By a modification of Lukasiewicz symbolism, we can write these conditions as

1 $EEEprEqpErq,$

2 $EEpEqrEEpqr,$

where E is the truth functor (for example, see A. N. Prior [2]). By this symbol, the axioms of the usual equivalence relation are considered as Epp, $EEpqEqp$, and $EEpqEEqrEpr$.

In the equivalential calculus, we use the rule of usual substitution and the rule of detachment: α and $E\alpha\beta$ imply β. By these rules, S. Leśniewski proved many theses of the equivalential calculus (see [1]).

In this note, we shall use prooflines by Lukasiewicz for the proof of theses and some metatheorems given below.

Assume that the conditions 1 and 2 hold, then

2 p/r—3,

3 $EErEqrEErqr.$

1 $p/r,\ q/Erq,\ r/Eqr$ $^*C3$—4,

4 $EEqrErq,$

which is a commutative law.

4 $q/EEprEqp,\ r/Erq$ $^*C1$—5,

5 $EErqEEprEqp.$

1 r/q $^*C4$ $q/p,\ r/q$—6,

6 $Eqq.$

Next we shall give some metatheorems on the equivalential calculus under the conditions 1 and 2. By the thesis 4, we have

A) *If* $E\alpha\beta$, *then* $E\beta\alpha$.

From the thesis 5 and A),

B) *If* $E\alpha\beta$, *then* $EE\gamma\beta E\alpha\gamma$, $EE\alpha\gamma E\gamma\beta$, $EE\gamma\alpha E\beta\gamma$, *and* $EE\beta\gamma E\gamma\alpha$.

To prove $EEpqEEqrEpr$, we use the metatheorem B). By the thesis 4, we have $EErpEpr$. From the first result of B), we have

$$EEEqrEprEErpEqr.$$

We use the first result of B), then

$$EEEpqEErpEqrEEEqrEprEpq.$$

From the thesis 5, we have

$$EEEqrEprEpq,$$

hence, by using the thesis 4,

7 $EEpqEEqrEpr$.

Therefore *we have* Epp, $EEpqEqp$, *and* $EEpqEEqrEpr$ *in the equivalential calculus.*

In a later paper, we shall prove these theses characterize the equivalential calculus.

Theorem 1. *Under the two rules of substitution and detachment,*

1) $EEpqEqp$, $EEpqEEqrEpr$ *imply* Epp,

2) $EEpqEqp$, $EEpqEEqrErp$ *imply* Epp,

3) Epp, $EEpqEEqrErp$ *imply* $EEpqEqp$.

Proof of 1). It is evident that

$$EEpqEEqrEprEEEqrEprEpq,$$

hence

8 $EEEqrEprEpq$,

8 q/p, r/p $*C4$ q/p, r/p—6,

6 Epp.

Proof of 2). First we have

9 $EEEqrErpEpq$,

9 q/p $*C4$ p/p—6,

6 Epp.

Proof of 3). We have

$$EEppEEprErp,$$

then by Epp, we have

$$EEprErp.$$

Theorem 2. $EEpqEErqEpr$ *implies* Epp *and* $EErqEqr$.

Proof. Let

1 $EEpqEErqEpr$.

1 p/Epq, $q/EErqEpr$, r/s $*C1$—2,

2 $EEsEErqEprEEpqs$.

2 s/Epq $*C1$—3,

3 $EEpqEpq$.

1 p/Epq, q/Epq *$C3$—4,

4 $EErEpqEEpqr$.

4 p/Erq, q/Epr, r/Epq *$C1$—5,

5 $EEErqEprEpq$.

5 q/p, r/p *$C3$ q/p—6,

6 Epp.

1 q/p *$C6$—7,

7 $EErqEqr$,

which completes the proof.

Theorem 3. *$EEpqEEprErq$ implies Epp and $EEprErp$.*

Proof. Let

1 $EEpqEEprErq$.

Then we have the following theses.

1 p/Epq, $q/EEpsEsq$ *$C1$ r/s—2,

2 $EEEpqrErEEpsEsq$.

2 p/Epq, q/r, $r/ErEEpsEsq$ *$C2$—3,

3 $EErEEpsEsqEEEpqsEsr$.

3 r/Epq *$C1$ r/s—4,

4 $EEEpqsEsEpq$.

4 $s/EEprErq$ *$C1$—5,

5 $EEEprErqEpq$.

2 p/Epr, q/Erq, r/Epq *$C5$—6,

6 $EEpqEEEprsEsErq$.

4 $s/EEEprsEsErq$ *$C6$—7,

7 $EEEEprsEsErqEpq$.

7 q/p, r/p, s/p *$C4$ q/p, s/p—8,

8 Epp.

1 q/p *$C8$—9,

9 $EEprErp$.

Theorem 4. *Under the rules of substitution and detachment, the following theses are equivalent:*

1 Epp, $EEpqEEqrErp$.

2 $EEpqErp$, $EEpqEErqErp$,

3 $EEpqEqp$, $EEpqEEqrEpr$.

4 $EEpqEErqEpr$.

5 $EEpqEErqEpr$.

Remark. As shown in a later paper by S. Tanaka, these theses characterize the equivalent calculus.

The proof of Theorem 4 is easy, for example, assume thesis 4:

1 $EEpqEErqEpr$.

1 p/Epq, $q/EErqEpr$, $r/EEprErq$ *$C1$—$C7$ in Theorem 2 q/Erq, r/Epr—2,

2 $EEpqEEprErq$,

which is thesis 5. By the similar techniques, we have Theorem 4.

Further, *EEpqEErpEqr* is equivalent to the axioms 1 and 2, which will be shown in a later paper by Y. Arai. As easily seen, the first two propositions in Theorem 1 hold under the rule of substitution and the rule of reverse detachment: $E\alpha\beta$ and β imply α.

References

[1] S. Leśniewski: Grundzüge eines neuen Systems der Grundlagen der Mathematik. Fund. Math., **14**, 1-81 (1929).

[2] A. N. Prior: Formal Logic. Oxford (1962).