70. On Fourier Series with Gaps[*]

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1. **Introduction.** Let $\{n_k\}(k=1,2,\cdots)$ be a strictly increasing sequence of positive integers. Let $f(x)$ be a real function, L-integrable over $(-\pi,\pi)$ and having a period 2π, whose Fourier coefficients a_n, b_n vanish except for $n=n_k$. Namely

$$(1)\qquad f(x)\sim\sum_{k=1}^{\infty}(a_{n_k}\cos n_kx+b_{n_k}\sin n_kx),$$

supposing for simplicity that the constant term also vanishes.

Let x_0 be a fixed point and $\alpha>0$, we write $f\in\mathrm{Lip}\,\alpha(P)$ if

$$|f(x_0+h)-f(x_0)|\leqq A\,|h|^{\alpha}$$

holds for all small h. We assume throughout that "A" denotes an absolute constant and two A's might be not equal even in the same equation for the sake of conveniency. M. and S. Izumi [1] proved the following

Theorem A. *If f has the Fourier series* (1) *with the gap condition*

$$(G_1)\qquad n_{k+1}-n_k\geqq An_k^{\beta}\qquad(0<\beta\leqq 1)$$

and $f\in\mathrm{Lip}\,\alpha(P)$, $(\alpha>0)$, then

$$a_{n_k},\ b_{n_k}=O(n_k^{-\alpha\beta})$$

Theorem B. *If f has the F.s.* (1) *with the Hadamard gap*

$$(G_2)\qquad n_{k+1}/n_k\geqq\lambda>1$$

and $f\in\mathrm{Lip}\,\alpha(P)$, $(0<\alpha<1)$, then f belongs to $\mathrm{Lip}\,\alpha$ class in $(-\pi,\pi)$.

Using the Izumis' method in Theorem A, we shall prove a group of theorems under a general gap condition[1),2)]

$$(G)\qquad n_{k+1}-n_k\geqq AF(n_k),\ n_k-n_{k-1}\geqq AF(n_k)$$

where $F(n_k)\uparrow\infty$ as $k\uparrow\infty$ and $F(n_k)\leqq n_k$ for all k.

Theorem 1. *If f has the F.s.* (1) *with the gap* (G) *and $f\in\mathrm{Lip}\,\alpha(P)$, $(\alpha>0)$, then*

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1) If F satisfies some regularity condition, for example, $F(n_k/2)>AF(n_k)$, then (G) may be replaced by

$$(G')\qquad n_{k+1}-n_k\geqq AF(n_k).$$

For if $n_k/n_{k-1}\leqq 2$, say, then by a suitable choice of A, we have

$$n_k-n_{k-1}\geqq AF(n_{k-1})\geqq AF(n_k/2)\geqq AF(n_k).$$

2) "A" may be included in F, but this form is convenient for later use.

$$a_{n_k}, b_{n_k} = O(F(n_k)^{-\alpha}).$$

In order to have a similar estimation as in Theorem B for a weaker gap condition, we now consider a new gap

$$(G) \qquad n_{k+1} - n_k \geqq A n_k^{\beta} k^{\gamma} \qquad (0<\beta<1, \gamma>0).$$

This is weaker than(G_2), but stronger than (G_1). A simple example of this kind of gap is $(k^{[\log k]})_{k=1,2,\dots}$.

We first derive a theorem concerning the absolute convergence.

Theorem 2. *If f has the F.s.* (1) *with the gap* (G_3) *and* $f \in \mathrm{Lip}\,\alpha(P)$, $(\alpha>0)$, *then* (1) *converges absolutely when* $\alpha\beta + \alpha\gamma + \beta > 1$.

Supposing $\gamma=0$ throughout the proof, (G_3) becomes (G_1), we can easily get the following

Corollary.[3] *If f has the F.s.* (1) *with gap* (G_1) *and* $f \in \mathrm{Lip}\,\alpha(P)$, $(\alpha>0)$, *then* (1) *converges absolutely when* $\alpha\beta + \beta > 1$.

Finally we shall prove the following

Theorem 3. *If f has the F.s.* (1) *with the gap* (G_3) *and* $f \in \mathrm{Lip}\,\alpha(P)$, $(\alpha>0)$, *putting* $\gamma = 1/\alpha$, *then*

(i) *f belongs to the* $\mathrm{Lip}\,\alpha\beta$ *class in* $(-\pi, \pi)$ *when* $\alpha\beta<1$;

(ii) *f belongs to the* $\mathrm{Lip}\,\delta$ *class in* $(-\pi, \pi)$, *for any* $\delta<1$, *when* $\alpha\beta=1$;[4]

(iii) *f belongs to the* $\mathrm{Lip}\,1$ *class in* $(-\pi, \pi)$ *when* $\alpha\beta>1$.

2. Proof of Theorem 1. a) $0<\alpha<1$. We can suppose that $x_0=0$. Let c_{n_k} be the n_k-th complex Fourier coefficient of f, then

$$c_{n_k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) T_{M_k}(x) e^{-in_k x} dx,$$

where $T_{M_k}(x)$ is a trigonometrical polynomial of degree $M_k = AF(n_k)$ and with constant term 1. Now

$$\begin{aligned} c_{n_k} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) T_{M_k}(x) e^{-in_k x} dx \\ &= \frac{-1}{2\pi}\int_{-\pi}^{\pi} f\left(x+\frac{\pi}{n_k}\right) T_{M_k}\left(x+\frac{\pi}{n_k}\right) e^{-in_k x} dx \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \left[f(x)T_{M_k}(x) - f\left(x+\frac{\pi}{n_k}\right) T_{M_k}\left(x+\frac{\pi}{n_k}\right)\right] e^{-in_k x} dx \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \left[f(x) - f\left(x+\frac{\pi}{n_k}\right)\right] T_{M_k}(x) e^{-in_k x} dx \\ &\quad + \frac{1}{4\pi}\int_{-\pi}^{\pi} f\left(x+\frac{\pi}{n_k}\right)\left[T_{M_k}(x) - T_{M_k}\left(x+\frac{\pi}{n_k}\right)\right] e^{-in_k x} dx \\ &= I + J, \text{ say.} \end{aligned}$$

Since the Fourier exponents of $f(x+\pi/n_k)$ with non-vanishing Fourier coefficients are the same as that of $f(x)$ and the trigonometrical

3) This corollary appeared as part of Theorem 2 in [1].

4) We really proved that $f(x+h)-f(x)=o(|h|^{\delta})$, and then $f \in \lambda_{\delta}$, by the notation in [2].

polynomial $T_{M_k}(x)-T_{M_k}(x+\pi/n_k)$ is of degree not exceeding M_k and with the constant term 0, we have $J=0$. We take $T_{M_k}(x)=2K_{M_k}(x)$, where $K_{M_k}(x)$ is the Fejér kernel of order M_k. Then we get

$$|T_{M_k}(x)|=\frac{\sin^2(M_k+1)\frac{1}{2}x}{(M_k+1)\sin^2\frac{1}{2}x}\leqq AM_k \quad\text{and}\quad |T_{M_k}(x)|\leqq\frac{A}{M_kx^2}.$$

Now

$$\begin{aligned} c_{n_k}=I&=\frac{1}{4\pi}\int_{-\pi}^{\pi}[f(x)-f(x+\pi/n_k)]\,T_{M_k}(x)e^{-in_kx}dx\\ &=\frac{1}{4\pi}\left(\int_{-1/M_k}^{1/M_k}+\int_{1/M_k}^{\pi}+\int_{-\pi}^{-1/M_k}\right)[f(x)-f(x+\pi/n_k)]\,T_{M_k}(x)e^{-in_kx}dx\\ &=I_1+I_2+I_3,\ \text{say,}\end{aligned}$$

where

$$|I_1|\leqq AM_k\int_{-1/M_k}^{1/M_k}|f(x)-f(x+\pi/n_k)|\,dx\leqq AM_k^{-\alpha}=O(F(n_k)^{-\alpha})$$

and

$$\begin{aligned}|I_2|&\leqq AM_k^{-1}\int_{1/M_k}^{\pi}|f(x)-f(x+\pi/n_k)|\frac{dx}{x^2}\\ &\leqq AM_k^{-1}\int_{1/M_k}^{\pi}x^{\alpha-2}dx\leqq AM_k^{-\alpha}=O(F(n_k)^{-\alpha}).\end{aligned}$$

Similarly we can get $|I_3|=O(F(n_k)^{-\alpha})$. Therefore we have

$$a_{n_k},\ b_{n_k}=O(F(n_k)^{-\alpha}).$$

b) $\alpha\geqq 1$. In this case we use the trigonometrical polynomial

$$T_{M_k}(x)=(2K_{[M_k/p]}(x))^p\Big/\int_{-\pi}^{\pi}(2K_{[M_k/p]}(x)^pdx,$$

instead of Fejér kernel, then we have

$$|T_{M_k}(x)|\leqq AM_k \quad\text{and}\quad |T_{M_k}(x)|\leqq AM_k^{1-2p}x^{-2p}.$$

Therefore, in the estimation of I_1 and I_2, α may be greater than or equal to 1. Thus the theorem holds also for $\alpha\geqq 1$.

3. Proof of Theorem 2. We shall prove first that $n_j\geqq j^\delta$ for all sufficient large j and $\delta\leqq\frac{\gamma+1}{1-\beta}$. Suppose $n_k\geqq k^\delta$ for some $k\geqq k_0$, then

$$n_{k+1}\geqq n_k+An_k^\beta k^\gamma\geqq k^\delta+Ak^{\delta\beta+\gamma},$$
$$(k+1)^\delta=k^\delta+\delta k^{\delta-1}+\cdots.$$

We have $n_{k+1}\geqq(k+1)^\delta$ since $\alpha\beta+\gamma\geqq\delta-1$. Now, by Theorem 1, taking $F(n_k)=n_k^\beta k^\gamma$, we have[5]

$$\sum_{k=1}^{\infty}|c_{n_k}e^{in_kx}|\leqq A\sum_{k=1}^{\infty}n_k^{-\alpha\beta_k-\alpha\gamma}\leqq A\sum_{k=1}^{\infty}k^{-\alpha\beta\delta-\alpha\gamma}$$

which is finite when $\alpha\beta\delta+\alpha\gamma>1$. δ may be taken near to $\frac{\gamma+1}{1-\beta}$, therefore (1) converges absolutely when $\alpha\beta+\alpha\gamma+\beta>1$.

5) In this special case, the gap condition (G_3) implies, by a suitable change of A, $n_{k+1}-n_k\geqq An_k^\beta k^\gamma$, $n_k-n_{k-1}\geqq An_k^\beta k^\gamma$; c.f. [1].

4. Proof of Theorem 3. We need the following

Lemma. $\sum_{k=1}^{K} n_k^{1-\alpha\beta}k^{-1}=O(1)$ *if* $\alpha\beta>1$;

$=O(\log K)$ *if* $\alpha\beta=1$;

$=O(n_k^{1-\alpha\beta})$ *if* $\alpha\beta<1$.

Proof of this Lemma. a) $\alpha\beta>1$ and b) $\alpha\beta=1$ are trivial cases. c) $\alpha\beta<1$. We divide again into 3 cases according to the order relations between n_k and k.

i) $n_k^{1-\alpha\beta}\succ k$; In this case, $n_{j+1}^{1-\alpha\beta}(j+1)^{-1}>n_j^{1-\alpha\beta}j^{-1}$ for all j. Hence

$$\sum_{k=1}^{K} n_k^{1-\alpha\beta}k^{-1}\leqq Kn_K^{1-\alpha\beta}K^{-1}=O(n_K^{1-\alpha\beta}).$$

ii) $n_k^{1-\alpha\beta}\sim k$; In this case,

$$\sum_{k=1}^{K} n_k^{1-\alpha\beta}k^{-1}=O(K)=O(n_k^{1-\alpha\beta}).$$

iii) $n_k^{1-\alpha\beta}\prec k$; We may suppose $n_k^{1-\alpha\beta}\sim k^{\delta}$ for some $\delta<1$. Then

$$\sum_{k=1}^{K} n_k^{1-\alpha\beta}k^{-1}=O\Big(\sum_{k=1}^{K} k^{-(1-\delta)}\Big)=O(K^{1-(1-\delta)})=O(K^{\delta})=O(n_k^{1-\alpha\beta}).$$

Therefore our Lemma is proved.

Proof of Theorem 3. By the argument in the proof of Theorem 2, in the case $\gamma=1/\alpha$, we see that the series (1) converges uniformly to f, i.e.

$$f(x)=\sum_{k=1}^{\infty} c_{n_k}e^{in_kx}.$$

Now, by Theorem 1,[6]

$$|f(x+h)-f(x)|=\Big|\sum_{k=1}^{\infty} c_{n_k}e^{in_kx}(e^{in_kh}-1)\Big|$$
$$=\Big|\sum_{k=1}^{\infty}c_{n_k}e^{in_kx}e^{in_kh/2}2i\sin n_kh/2\Big|$$
$$\leqq 2\sum_{k=1}^{\infty}|c_{n_k}|\,|\sin n_kh/2|\leqq A\sum_{k=1}^{\infty}n_k^{-\alpha\beta}k^{-1}\,|\sin n_kh/2|.$$

If h is small, then there is a K such that $n_{K+1}^{-1}<|h|\leqq n_K^{-1}$ and

$$|f(x+h)-f(x)|\leqq A\Big(\sum_{k=1}^{K}+\sum_{k=K+1}^{\infty}\Big)n_k^{-\alpha\beta}k^{-1}\,|\sin n_kh/2|=A(S+T),\text{ say.}$$

We can suppose that, by some modifications,[7]

(2) $\qquad n_{2k}/n_k\geqq\lambda>1 \qquad$ for all k.

Then

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6) See the foot-note 5).

7) If not, we insert several terms between the n_k-th and n_{2k}-th terms of given Fourier series so that (2) holds and the coefficients of inserted terms are small enough. Then, following the same arguments, we have, $f+g$ has the desired result, where g is the sum of inserted terms which may be taken as differentiable. Therefore so is f.

$$
\begin{aligned}
|T| &\leqq \sum_{k=K+1}^{\infty} n_k^{-\alpha\beta} k^{-1} = \sum_{\nu=0}^{\infty} \sum_{k=2^\nu(K+1)}^{2^{\nu+1}(K+1)} n_k^{-\alpha\beta} k^{-1} \\
&\leqq \sum_{\nu=0}^{\infty} n_{2^\nu(K+1)}^{-\alpha\beta} [2^\nu(K+1)(2^\nu(K+1))^{-1}] = \sum_{\nu=0}^{\infty} n_{2^\nu(K+1)}^{-\alpha\beta} \\
&= n_{K+1}^{-\alpha\beta}\left[1 + \left(\frac{n_{2(K+1)}}{n_{K+1}}\right)^{-\alpha\beta} + \left(\frac{n_{2(K+1)}}{n_{K+1}}\right)^{-\alpha\beta}\left(\frac{n_{2^2(K+1)}}{n_{2(K+1)}}\right)^{-\alpha\beta} + \cdots\right] \\
&\leqq n_{K+1}^{-\alpha\beta}(1 + \lambda^{-\alpha\beta} + \lambda^{-2\alpha\beta} + \cdots) = O(n_{K+1}^{-\alpha\beta}) = O(|h|^{\alpha\beta}).
\end{aligned}
$$

By Lemma, we have:

i) For $\alpha\beta<1$;

$$|S| \leqq \tfrac{1}{2}|h| \sum_{k=1}^{K} n_k^{1-\alpha\beta} k^{-1} = O(|h|\, n_K^{1-\alpha\beta}) = O(|h|^{\alpha\beta}).$$

Hence $f \in \operatorname{Lip} \alpha\beta$ in $(-\pi, \pi)$.

ii) For $\alpha\beta=1$;

$$
\begin{aligned}
|S| &\leqq \tfrac{1}{2}|h| \sum_{k=1}^{K} n_k^{1-\alpha\beta} k^{-1} = O(|h| \log K) \\
&= O(|h|^{\delta} n_k^{-1+\delta} \log K) = O(|h|^{\delta}) \qquad \text{for any } \delta<1.
\end{aligned}
$$

Hence $f \in \operatorname{Lip} \delta$ in $(-\pi, \pi)$.

iii) For $\alpha\beta>1$;

$$|S| \leqq \tfrac{1}{2}|h| \sum_{k=1}^{K} n_k^{1-\alpha\beta} k^{-1} = O(|h|).$$

Hence $f \in \operatorname{Lip} 1$ in $(-\pi, \pi)$. Therefore the proof is completed.

References

[1] M. Izumi and S. Izumi: On lacunary Fourier series. Proc. Japan Acad., **41** (8), 648-651 (1965).

[2] A. Zygmund: Trigonometrical Series (2nd ed.), Vol. 1, Cambridge (1959).