101. *Some Theorems in B-Algebra. II*

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In our notes ([1]$\sim$[5] and [7]), we gave an algebraic formulations of two valued propositional calculus, and we proved some results in propositional calculus by the algebraic method. In this note, we shall prove two theses by J. Lukasiewicz mentioned in C. A. Meredith [6], as these proofs are not given in his paper.

Lukasiewicz theses are written in the forms of

CCCpqCCNrsCNttCCtpCuCrp,

CCCpqCCCNrNstrCuCCrpCsp.

To prove these, we shall use the algebraic method. In our notations,

Theorem 4. *In the B-algebra, we have*

$$((p*r)*u)*(p*t)\leqslant((t*\sim t)*(s*\sim r))*(q*p).$$

Proof. Let

(1) $((p*r)*u)*(p*t)\leqslant((t*\sim t)*(s*\sim r))*(q*p).$

we first mention some lemmas needed in the proof.

(2) $x*x=0$, i.e. $x\leqslant x$, ([2], p. 805).

(3) $x*(x*\sim y)\leqslant x*y$, ([3], p. 808).

(4) $x\leqslant y$ implies $x*z\leqslant y*z$, $z*y\leqslant z*x$, ([2], p. 806 or [3], p. 809).

(5) $(x*z)*(y*z)\leqslant(x*y)*z$, ([2], p. 805).

(6) $x*y=\sim y*\sim x$, ([2], p. 805 and p. 807).

(7) $(x*y)*z=(x*z)*y$, ([2], p. 808).

(8) $x*y\leqslant\sim y$, ([2], p. 806).

Next we shall prove (1). From (7), we have

$$((p*r)*u)*(p*t)=((p*r)*(p*t))*u.$$

To prove (1), by (7), it is sufficient to show

(9) $((p*r)*(p*t))*(((t*\sim t)*(s*\sim r))*(q*p))\leqslant u.$

Since u is arbitrary, if we prove

(10) $(p*r)*(p*t)\leqslant((t*\sim t)*(s*\sim r))*(q*p),$

then we have (9), and we complete the proof of (1).

By (2) and (3), we have

$$t*(t*\sim t)\leqslant t*t=0,$$

hence $t\leqslant t*\sim t$. By (3), we have

(11) $t*(s*\sim r)\leqslant(t*\sim t)*(s*\sim r).$

From (8), and $\sim(\sim x)=x$,

(12) $s*\sim r\leqslant t*(s*\sim r).$

(4) and (12) imply

(13) $t*r\leqslant t*(s*\sim r)$.

From (11) and (13), we have

(14) $t*r\leqslant(t*\sim r)*(s*\sim r)$.

(4) and (14) imply

(15) $(t*r)*(q*p)$
$\leqslant((t*\sim r)*(s*\sim r))*(q*p)$.

(4) and $q*p\leqslant\sim p$ obtained by (8) imply

(16) $(t*r)*\sim p\leqslant(t*r)*(q*p)$.

On the other hand, by (5) and (6), we have

(17) $(p*r)*(p*t)=(\sim r*\sim p)$
$*(\sim t*\sim p)\leqslant(\sim r*\sim t)*\sim p$
$=(t*r)*\sim p$.

(15), (16), and (17) imply

$$(p*r)*(p*t)\leqslant((t*\sim t)*(s*\sim r))*(q*p),$$

which (10). Hence we complete the proof of Theorem 4.

On the another formula, we have

Theorem 5. *In the B-algebra, we have*

$$((x*s)*(x*t))*u\leqslant(t*(z*(\sim s*\sim t)))*(q*p).$$

Proof. To prove this, we need some auxiliary formulas.

(18) $x*y\leqslant x$ ([3], p. 808).

By (8) and (6), we have

$$0=(x*y)*\sim y=y*\sim(x*y).$$

Hence

(19) $y\leqslant\sim(x*y)$.

From (18), (7), we have

(20) $((x*s)*(x*t))*u\leqslant(x*s)*(x*t)$
$=(x*(x*t))*s$.

Further from (3), (4), and (7), we have

(21) $(x*(x*t))*s\leqslant(x*\sim t)*s$
$=(t*\sim x)*s=(t*s)*\sim x$.

By (8) and (7),

(22) $(t*s)*\sim x\leqslant(t*s)*(y*x)$.

Then, by (20), (21), and (22), we have

(23) $((x*s)*(x*t))*u\leqslant(t*s)*(y*x)$.

On the other hand, $\sim s*\sim t=t*s$ implies that the right side of the formula to prove is written in the form of $(t*(z*(t*s)))*(y*x)$. Therefore by (23), it is sufficient to prove that

(24) $(t*s)*(y*x)\leqslant(t*(z*(t*s)))*(y*x)$.

$x*(x*\sim y)\leqslant x*y$ (H 5 in [3] or (3)) implies $x*(x*\sim x)\leqslant x*x=0$, then $x\leqslant x*\sim x$. Therefore we have

$$(t*s)*(y*x)\leqslant((t*\sim t)*s)*(y*x)$$
$$=((t*s)*\sim t)*(y*x).$$

By (19), we have $x*s \leqslant \sim(z*(x*s))$, therefore from the above formula, we have

$$(t*s)*(y*x) \leqslant ((t*s)*\sim t)*(y*x)$$
$$\leqslant (\sim(z*(t*s))*\sim t)*(y*x)$$
$$\leqslant (t*(z*(t*s)))*(y*x).$$

Consequently, we have (24), and then we complete the proof of Theorem 5.

References

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