100. On Axiom Systems of Propositional Calculi. XXI

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(Comm. by Kinjirô KUNUGI, M.J.A., May 12, 1966)

In this note, we shall prove theses of the implicational calculus mentioned in a paper of J. Lukasiewicz and A. Tarski [2]. The fundamental axioms of the implicational calculus are given by

1 $CpCqp$,

2 $CCpqCCqrCpr$,

3 $CCCpqpp$

and two usual rules of inference. In their paper [2], they state that the implicational calculus characterize by

4 $CCCpCqpCCCCCrstuCCsuCruvv$

or

5 $CCCpqCCrstCCuCCrstCCpuCst$.

In a joint paper by Y. Imai and the present writer [1], an algebraic formulation of the implicational calculus is given, and some useful formulas are proved by the pure algebraic technique.

To prove the above formulas, we shall use the algebraic method. By the I-algebra, we mean that an abstract algebra $M=\langle X, 0, *\rangle$ satisfying the following conditions:

I 1 $x*y\leqslant x$,

I 2 $(x*y)*(x*z)\leqslant z*y$,

I 3 $x\leqslant x*(y*x)$,

I 4 $0\leqslant x$,

I 5 $x*y=0$ if and only if $x\leqslant y$.

If $x\leqslant y$ and $y\leqslant x$, then we put $x=y$. By our notations, the formulas 4 and 5 are written in the form of

(6) $v\leqslant(v*(((u*r)*(u*s))*(u*(t*(s*r)))))*((p*q)*p)$,

(7) $((t*s)*(u*p))*((t*(s*r))*u)\leqslant(t*(s*r))*(q*p)$.

In [1], we prove the following formulas:

(8) The I-algebra is a partially ordered set on the relation.

(9) $x*0=x$, ((8) in [1]),

(10) $x*(x*y)\leqslant y$ ((11) in [1]),

(11) $(x*y)*z=(x*z)*y$, ((9) in [1]),

(12) $(x*y)*(z*y)\leqslant x*z$, ((13) in [1]),

(13) $(x*y)*(x*z)\leqslant x*(u*(z*y))$ ((31) in [1]).

We first prove (6). Axiom I 1 implies $(p*q)*p=0$, then (6) is equivalent to

(14) $v\leqslant v*(((u*r)*(u*s))*(u*(t*(s*r))))$

by (9). To prove (14), by (10), it is sufficient to show that

(15) $(u*r)*(u*s)\leqslant u*(t*(s*r))$.

(15) follows from (13). Hence we have (6).

Next we shall prove the formula (7). First, by I 1, $s*r\leqslant s$. Hence by I 2, we have $t*s\leqslant t*(s*r)$. Therefore by (13), we have

$$(t*s)*(u*p)\leqslant(t*(s*r))*(u*p).$$

Then the formula (13) implies

(16) $$((t*s)*(u*p))*((t*(s*r))*u) \leqslant((t*(s*r))*(u*p))*((t*(s*r))*u) \leqslant(t*(s*r))*(q*(u*(u*p))).$$

Now we use the following formula which is proved in [1]:

$$x*(x*y)=y*(y*x) \qquad ((23) \text{ in } [1]).$$

Then we have

$$q*(u*(u*p))=q*(p*(p*u)).$$

By I 1, we have $p*(p*u)\leqslant p$. Hence by I 2, $q*p\leqslant q*(u*(u*p))$. Therefore we have

(17) $(t*(s*r))*(q*(u*(u*p)))\leqslant(t*(s*r))*(q*p)$.

(16) and (17) implies

$$((t*s)*(u*p))*((t*(s*r))*u)\leqslant(t*(s*r))*(q*p),$$

which is the formula (7). Then we have the following

Theorem. *In the I-algebra, the following formulas hold*

(6) $v\leqslant(v*(((u*r)*(u*s))*(u*(t*(s*r)))))*((p*q)*p)$,

(7) $((t*s)*(u*p))*((t*(s*r))*u)\leqslant(t*(s*r))*(q*p)$.

The proof that the I-algebra is characterized by these formulas will be given in a later paper.

References

[1] Y. Imai and K. Iséki: On axiom systems of propositional calculi. XIV. Proc. Japan Acad., **42**, 19–22 (1966).

[2] J. Lukasiewicz and A. Tarski: Untersuchungen über den Aussagenkalkul. C. R. de Varsovie, Cl. III, **23**, 30–50 (1930).