131. On Smoothness of Normed Lattices

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1. Throughout this note, let $\boldsymbol{R}$ be a continuous semi-ordered linear space (i. e., conditionally σ-complete vector lattice) with a semi-continuous norm, i. e., $\|a_n\| \uparrow_{n=1}^{\infty}\|a\|$ if $0 \leqq a_n \uparrow_{n=1}^{\infty} a$ for a_n and a in $\boldsymbol{R}$.

For mutually orthogonal elements a and b in $\boldsymbol{R}$ with $\|a\|=\|b\|=1$, the curve $C(a, b)$: $\{(\xi, \eta);\ \|\xi a+\eta b\|=1\}$ is called an indicatrix of $\boldsymbol{R}$. $\boldsymbol{R}$ is said to be smooth, if at every point of the unit surface $\boldsymbol{S}$ of $\boldsymbol{R}$ there is only one supporting hyperplane of the unit sphere of $\boldsymbol{R}$, or equivalently the norm on $\boldsymbol{R}$ is differentiable in a sense of Gateaux.

In a normed space, it is well known that the smoothness of the space is equivalent to that of every two dimentional linear subspace. However, in a normed lattice, we shall be able to show that this fact yields from the properties of subspaces spanning by two orthogonal elements in the space.

The purpose of present note is to give a relation between the smoothness of R and the indicatrix of R, which supplement the investigations concerning the indicatrices by H. Nakano [4] and is used for an another material.

Theorem. *In order to* $\boldsymbol{R}$ *be smooth, it is necessary and sufficient that the norm on is continuous and every indicatrix of* $\boldsymbol{R}$ *has the tangent line at each point on the curve.*

To prove the theorem, we first state the lemmas.

Lemma 1 (T. Ando [1]). *If* $\boldsymbol{R}$ *is smooth, then* $\boldsymbol{R}$ *has a continuous norm.*[1]

x in $\boldsymbol{R}$ is called a *simple element with respect to* a non-zero element a in $\boldsymbol{R}$, if x has a form: $x=\sum_{i=1}^{n}\alpha_i[p_i]$ a where $\{[p_i];\ 1\leqq i\leqq n\}$ is of mutually orthogonal projectors[2] in $\boldsymbol{R}$.

Lemma 2. *When* $\boldsymbol{R}$ *has a continuous norm, for any* $0\neq a\in \boldsymbol{R}$ *and* $0\leqq x\in \boldsymbol{R}$ *there exists a squence of simple elements* x_n *with respect to* a *such that*

$$0\text{-}\lim_{n\to\infty} x_n=[a]x. \qquad (0\text{-lim } \textit{means the order limit.})$$

Proof. First let $a>0$. In accordance with [5], let $\mathcal{E}$ be the

1) The norm $\|\cdot\|$ on $\boldsymbol{R}$ is called a *continuous norm* if $\boldsymbol{R}\ni a_n \downarrow_{n=1}^{\infty} 0$ implies $\|a_n\| \downarrow_{n=1}^{\infty} 0$.

2) See H. Nakano [5; § 6].

proper space of $\boldsymbol{R}$ and $(x/a, \mathscr{P})$ be the relative spectrum which is necessarily almost finite and continuous function in the neighborhood $U_{[a]}$ in $\mathcal{E}$. For any positive integer n

$$A_{n,i}=\left\{\mathscr{P};\ \frac{i-1}{2^n}<\left(\frac{x}{a}, \mathscr{P}\right)<\frac{i}{2^n}\right\}^{-} \qquad (i=1, 2, \cdots, n2^n)$$

are both open and closed in $\mathcal{E}$, since $\mathcal{E}$ is σ-universal, therefore there exist mutually orthogonal projectors $[p_{n,i}]\leqq[a]$ such that $U_{[p_{n,i}]}=A_{n,i}$ (i$=1, 2, \cdots, n2^n$). Next, putting $x_n=\sum_{i=1}^{n\cdot 2^n}\{(i-1)/2^n\}[p_{n,i}]a$, we obtain, by Theorem 16.7 in [5], 0-$\lim_{n\to\infty} x_n=[a]x$. If a is non-positive, for the positive part a^+ and the negative part a^- of a, we have sequences $\{x_n^+\}$ and $\{x_n^-\}$ of simple elements with respect to a^+ and a^- respectively such that 0-$\lim_{n\to\infty}x_n^+=[a^+]x$ and 0-$\lim_{n\to\infty}x_n^-=[a^-]x$. Therefore $x_n=x_n^+-x_n^-$ satisfy the required relation.

The proof of Theorem. Necessity. Suppose $\boldsymbol{R}$ is smooth. Then, for any $x\in \boldsymbol{S}$ and any projecto $[p]$, the Gateaux's differential

$$G(x;\ [p]x)=\lim_{\varepsilon\to 0}\frac{\|x+\varepsilon[p]x\|-\|x\|}{\varepsilon}$$

exists. Since the indicatrix $C(a, b)$ of $\boldsymbol{R}$ is symmetric in respect to the axices of two dimensional Euclidean space, we consider the function $\eta=\eta(\xi)$ expressing $C(a, b)$ restricted on the upper half-plane. In $|\xi|<1$, the function $\eta(\xi)$ is obviously one-valued, continuous, concave function. As is shown in the proof of Lemma 2 in [3], $\eta(\xi)$ is differentiable at each ξ, $|\xi|<1$, except for $\xi=0$. It is evident, that $[d\eta/d\xi]_{\xi=0}=0$ from $G(x;\ 0)=0$. Accordingly, the indicatrix $C(a, b)$ has a tangent line on the curve confined for $|\xi|<1$. When we replace the variables ξ and η, it follows easily that $C(a, b)$ has a tangent line at each vertex (i. e., $(\pm 1, 0)$ and $(0, \pm 1)$) on the curve provided that $C(a, b)$ does not pass through the point $(1, 1)$. However, if $C(a, b)$ includes the point $(1, 1)$, then $\|a+b\|=\|a\|=\|b\|=1$ and

$$\lim_{\varepsilon\to+0}\frac{\|a+(1+\varepsilon)b\|-1}{\varepsilon}=1, \qquad \text{but} \qquad \lim_{\varepsilon\to-0}\frac{\|a+(1+\varepsilon)b\|-1}{\varepsilon}=0,$$

namely, $G(a\,;b)$ does not exist. This contradicts to the smoothness of $\boldsymbol{R}$. Thus, by Lemma 1 the necessity is proved.

Sufficiency. We denote by $G^+(x\,;\cdot)$ and $G^-(x\,;\cdot)$ the Gateaux's right and left derivatives at $x\in \boldsymbol{S}$ respectively. For a indicatrix $C(a, b)$, let $\eta=\eta(\xi)$ be the same as the function used in the proof of necessity. Then $\eta(\xi)$ is differentiable at each ξ, $|\xi|<1$. Therefore, for each ξ, $|\xi|<1$, and arbitrarily small $\varepsilon>0$, we have $\eta(\xi\pm\varepsilon)=\eta(\xi)\pm\varepsilon\{\eta'(\xi)+h(\pm\varepsilon)\}$ where $h(\pm\varepsilon)\to 0$ as $\varepsilon\to+0$. Therefore, by the convexity of the norm and definition of $\eta(\xi)$ it is seen that

$$G^+(\xi a+\eta(\xi)b;\ a+\eta'(\xi)b)\leqq 0\leqq G^-(\xi a+\eta(\xi)b;\ a+\eta'(\xi)b)$$

and hence

(1) $$G(\xi a+\eta(\xi)b;\ a+\eta'(\xi)b)=0.$$

As is well known, for $x\in \boldsymbol{S}$ if $G(x\,;y)$ and $G(x\,;z)$ exist for y, $z\in R$, then

(2) $$G(x\,;\alpha y+\beta z)=\alpha G(x\,;y)+\beta G(x\,;z)$$

for any real numbers α and β. Accordingly, from $G(x\,;x)=1$ for any $x\in \boldsymbol{S}$, the relation (1) implies $G(\xi a+\eta(\xi)b;(\eta(\xi)-\xi\cdot\eta'(\xi))b)=1$ and consequently $G(\xi a+\eta b;[b](\xi a+\eta b))$ exists. Similary, $G(\xi a+\eta b;[a](\xi a+\eta b))$ exists.

Next, take $x\in \boldsymbol{S}$ and a projector $[p]$. In the cases when $[p]x=0$ or $[p]x=x$, we have $G(x;[p]x)=0$ or 1. Hence we consider the case when $[p]x\neq 0$ and $(1-[p])x\neq 0$. If $\|[p]x\|<1$ or $\|(1-[p])x\|<1$, then setting $a=[p]x/\|[p]x\|$ and $b=(1-[p])x/\|(1-[p])x\|$ the existence of $G(x;[p]x)$ is shown by preceding argument. And also $\|[p]x\|=\|(1-[p])x\|=1$ not arise. Indeed, if it is possible, the indicatrix $C([p]x/\|[p]x\|,(1-[p])x/\|(1-[p])x\|)$ includes the point (1, 1) so that this indicatrix has not the tangent line at this point, which contradicts to the Assumption. Thus, for any $x\in \boldsymbol{S}$ and a projector $[p]$, we could show to existence of $G(x;[p]x)$.

To prove the smoothness of $\boldsymbol{R}$, fixed $a\in \boldsymbol{S}$, for any $0\leqq x\in \boldsymbol{R}$ there exist simple elements x_n with respect to a for which satisfy $0\text{-}\lim x_n=[a]x$ by Lemma 2. Supposing $x=\sum_{i\in I}\alpha_i[p_i]$ a where I is a finite index set, we abtain

$$G(a;x_n)=\sum_{i\in I}\alpha_i G(a;[p_i]a)$$

from the above discussion. On account of $|G(a;x_n)|\leqq\|x_n\|\leqq\|[a]x\|$, we can define, by the continuity of the norm,

(3) $$F_a(x)=\lim_{n\to\infty}G(a;x_n).$$

This value depends only on x and a in $\boldsymbol{R}$. Indeed, for $0\leqq x\in \boldsymbol{R}$ and $a\in \boldsymbol{S}$ taking two system $\{x_n\}$ and $\{y_n\}$ sucn that $0\text{-}\lim x_n=[a]x$ and $0\text{-}\lim y_n=[a]x$ by Lemma 2, we have $|G(a;x_n)-G(a;y_n)|\leqq\|x_n-y_n\|\to 0$ as $n\to\infty$.

Next, for any $x\in \boldsymbol{R}$ we define a functional $F(a;x)$ on $\boldsymbol{R}$ as $F(a;x)=F_a(x^+)-F_a(x^-)$. Then, $F(a;x)$ is a linear functional on $\boldsymbol{R}$ by (2) and has the properties: $|F(a;x)|\leqq\|x\|$ for every x in $\boldsymbol{R}$ and $F(a;x)=0$ for all $[a]x=0$.

On the other hand, for a supporting hyperplane of the unit sphere at a, denote it by $\{x;(x,\bar{a})\}=1$ for some $\bar{a}\in \overline{\boldsymbol{R}}$ with $\|\bar{a}\|=1$, with $\|\bar{a}\|=1$, where $\overline{\boldsymbol{R}}$ is the adjoint space of $\boldsymbol{R}$, it is known that $G^-(a;x)\leqq(x;a)\leqq G^+(a;x)$ for every $x\in \boldsymbol{R}$. Hence, from (3) we obtain $(x,a)=F(a;x)$. Thus the uniqueness of the supporting hyperplane of the unit sphere at $a\in \boldsymbol{S}$ is shown. The proof of the theorem is completed.

2. In a finite modulared semi-ordered linear space $\boldsymbol{R}$ with a modular m, it follows that $m(x)=1$ is equivalent to $||x||=1$ where $||\cdot||$ is the modular norm. (See [5] for definitions) Then, since an indicatrix $C(a, b)$ of $\boldsymbol{R}$ is represented by $\{(\xi, \eta): m(\xi a)+m(\eta b)=1\}$, we have the following:

Corollary 1. [2; Theorem 2.4] *If a modular m is finite and even, then its modular norm is smooth. In a case when $\boldsymbol{R}$ is non-atomic, the inverse statement is also true.*

Proof. The first assertion is evident from Theorem. The inverse statement follows from Lemma 1 and that the continuity of the modular norm implies the finiteness of the modular in a non-atomic case.

Moreover, M. Rao [6] had discussed on the smoothness of the Orlicz space on a measure space without assumptions of non-atomicity and σ-finiteness. In regard to his work, we have also the following:

Corollary 2. *If a Orlicz space $\boldsymbol{L}_{\Phi}(\Omega, \mu)$, associated with N-function satisfies the following conditions, then the Luxemburg norm $||\cdot||$ is smooth.*

i) *$\Phi(2\xi)\leqq K\Phi(\xi)$ for some $K>0$ and every real $\xi>0$,*

ii) *Φ has the continuous derivative φ.*

Proof. For any $x\in \boldsymbol{L}_{\Phi}$ with $||x||=1$, we have

$$\int_{\Omega}|x(t)|\,\varphi(|x(t)|)\,d\mu\leqq\int_{\Omega}\Phi(2|x(t)|)d\mu$$

and hence

$$\frac{d}{d\xi}\int_{\Omega}\Phi(\xi|x(t)|)\,d\mu=\int_{\Omega}|x(t)|\varphi(\xi|x(t)|)d\mu.$$

Therefore, any indicatrix $C(|a|, |b|)$:

$$\left\{(\xi, \eta); \int_{\Omega}\Phi(\xi|a(t)|)\,d\mu+\int_{\Omega}\Phi(\eta|b(t)|)d\mu=1\right\}$$

has a tangent line at each point on the curve by C and hence the norm is smooth by Theorem.

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