160. On P^* Spaces and Equicontinuity

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Let P be a topological property.[1] A topological space X is called a *P^* space* if a subset U of X is open in X whenever $U\cap K$ is open in K for any subset K in X satisfying P. The purpose of this note is to investigate properties of P^* spaces and as applications to obtain some extensions of a theorem of Gleason [2] and the Ascoli's theorem.

1. Let E be a set, then we can consider the lattice of all topologies on E, that is, the ordering of the lattice can be defined as follows; $X\geqq Y$ if $O(X)\supset O(Y)$, where $O(X)$ (or $O(Y)$) is the set of all open subsets in X (or Y). For any family $\{X_j\}$ of topological spaces on E, $\vee X_j$ or $\wedge X_j$ denotes the join or the meet of $\{X_j\}$ ([4], [6]). A topological property P is said to have the *condition* (C) if it satisfies the following condition; any space consisting of one point has P, and any continuous image of X also satisfies P if a topological space X has P. Examples of topological properties having (C) are "compact", "separable", "connected", and "arcwise connected",[2] and any k-space ([5]) is a P^* space, where P is "compact".

We first prove the following theorem.

1.1. Theorem. *Let a topological property P have (C). If $\{X_\alpha\}$ are P^* spaces on the same basic set, then $\wedge X_\alpha$ is also a P^* space.*

Proof. Put $Z=\wedge X_\alpha$, then Z is a quotient space (cf. [5]) of $\sum X_\alpha$, where $\sum X_\alpha$ denotes the sum of X_α.[3] Since $\{X_\alpha\}$ are P^* spaces, it is clear that $\sum X_\alpha$ is a P^* space, so by the next lemma, the theorem is proved.

1.2. Lemma. *Let P be a topological property satisfying (C). If X is a P^* space then any quotient space of X is also a P^* space.*

Proof. The lemma can be proved easily.

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1) Let P be a property of topological spaces. P is said to be topological if it is invariant under homeomorphisms.

2) X is arcwise connected if for two points a, b in X there is a continuous image of closed interval containing a, b in X.

3) The fact is due to Professor K. Morita. In $\sum X_\alpha$, $\{X_\alpha\}$ are mutually disjoint and any X_α is open in $\sum X_\alpha$.

Let P be a topological property. A topological space X is called a *w-locally P space* (or *a locally P space*) if for any point x in X, there is a neighborhood (or an arbitrarily small neighborhood) having P. It is clear that a locally P space is always a w-locally P space, but the converse is false in general (for example, if P is "connected").

1.3. Theorem. *Any P^* space is a meet of w-locally P spaces.*

Proof. Let X be a P^* space and let $\mathfrak{K}$ be the family of subsets in X satisfying P. For any $K \in \mathfrak{K}$, we define a topological space X_K as follows: $O(X_K)=\{(K\cap U)\cup M;\ U \in O(X)$ and M is any subset which is disjoint from $K\}$. Since K has P in X_K and any space consisting of one point satisfies P, X_K is a w-locally P space. In order to prove that $X=\wedge X_K$, we need only to show that $\wedge X_K \leqq X$, because it is clear that $\wedge X_K \geqq X$. If $W \in O(\wedge X_K)$, then $W \in O(X_K)$ for any K, so W is of form $(K\cap U_K)\cup M_K$, where U_K is an open set in X and $M_K \cap K=\phi$. $W\cap K=K\cap U_K$, and it is open in K. Since X is a P^* space, W is open in X.

1.4. Corollary. *Any P^* space is a quotient space of a w-locally P space.*

Let P be a topological property, then P^* can be also regarded as a topological property, so we can define *P^{**} spaces* as $(P^*)^*$ spaces.

1.5. Theorem. *If P has (C), P^{**} spaces coincide with P^* spaces.*

Proof. It is clear that a P space (=a space having P) is a P^* space for any topological property P, so any P^* space is a P^{**} space. We need only to prove that any P^{**} space is a P^* space. Let X be a P^{**} space, then X is a quotient space of a w-locally P^* space by the above corollary. On the other hand, by next lemma, any w-locally P^* space is a P^* space. Therefore X is a quotient space of a P^* space. By Lemma 1.2, X is a P^* space.

1.6. Lemma. *Any w-locally P^* space is a P^* space.*

Proof. The lemma can be proved easily.

A topological property P is said to have the *condition* (N) if for any family of subsets $\{A_\alpha\}$ in a topological space such that any A_α has P and $\cap A_\alpha \neq \phi$, $\cup A_\alpha$ has P. The topological property "connected" or "arcwise connected" has (N).

1.7. Theorem. *If P has (N), P^* spaces coincide with w-locally P spaces.*

Proof. It is clear that a w-locally P space is a P^* space. To prove the converse, let X be a P^* space. Then, by (N), there is the largest P space C_x in X containing x for any $x \in X$. In order to prove the theorem, it suffices to show that C_x is open for any $x \in X$. For any K having P, (1) if $C_x \cap K \neq \phi$, $C_x \cup K$ is also a P space by (N). Since C_x is the largest P space, $C_x \supset K$, so $C_x \cap K = K$ is open in K, (2) if $C_x \cap K = \phi$, $C_x \cap K$ is, of course, open in K. Since X is a P^* space, C_x is open.

Remark. In Theorem 1.7, that P has (N) is not a necessary condition in order that P^* spaces coincide with w-locally P spaces: If P has (C) and if $Q = P^*$, then Q^* spaces coincide with w-locally Q spaces (they are also equal to Q spaces, cf., Theorem 1.5).

1.8. Theorem. *Suppose that P has (C), then any meet of locally P^* spaces is also a locally P^* space.*

Proof. Let $\{X_\alpha\}$ be locally P^* spaces. Let $X = \wedge X_\alpha$ and let O be an open subset in X. Since O is open in X_α, the restriction Y_α of X_α to O is a w-locally P^* space, so it is a P^* space. By Theorem 1.1, $Y = \wedge Y_\alpha$ is a P^* space. Since it is clear that Y is the restriction of X to O, X is a locally P^* space.

2. Gleason [2] has proved the following theorem: let S be a topological space. Then there exists a locally connected topological space S^* and a continuous one-to-one mapping φ of S^* onto S such that if f is any continuous mapping of a locally connected space A into S, then f can be factored in the form $f = \varphi \circ f^*$ where f^* is a continuous mapping of A into S^*. Since if P is "connected", locally P^* spaces coincide with locally connected spaces, the following theorem is an extension of the theorem of Gleason.[4)]

2.1. Theorem. *Let P have (C) and let S be a topological space. Then there exists a locally P^* space S^* and a continuous one-to-one mapping φ of S^* onto S such that if f is any continuous mapping of a locally P^* space A into S, then $f = \varphi \circ f^*$ where f^* is a continuous mapping of A into S^*.*

Proof. Put $S^* = \wedge\{S_\alpha; S_\alpha$ is a locally P^* space and $S_\alpha \geqq S\}$. By Theorem 1.8, S^* is also a locally P^* space. Let f be any continuous mapping of A into S, then we define a topological space T as follows; $O(T) = \{W; f^{-1}(W) \in O(A)\}$. We see that T is a locally P^* space. For, let W be any open set in T, then $f^{-1}(W) = U \in O(A)$. Since A is a locally P^* space, U is a w-locally P^* space, so U is a P^* space (Lemma 1.6). We can here regard that W is a quotient space of U. By Lemma 1.2, W is also a P^* space, so T is a locally P^* space. Now since f is continuous mapping of A into T and $T \geqq S^*$, the theorem is proved.

4) The similar theorem was proved in [3] for locally arcwise connected spaces. If P is "arcwise connected", locally P^* spaces coincide with locally arcwise connected spaces. In [2] Gleason discussed a generalization of his theorem, but our method is a little different from his.

3. Let P be a topological property. A topological space X is called a *P_0^* space* if a subset $U(\ni x_0)$ in X is a neighborhood of x_0 in X, whenever $U \cap K$ is a neighborhood of x_0 in K for any $K(\ni x_0)$ satisfying P. A neighborhood here need not be an open set. It is clear that a P_0^* space is a P^* space. If P is "compact", P_0^* spaces equal to k_0 spaces [7]. Let X_j be a family of topological spaces on the same set E. Then we define a space X on E as follows: $V(x)$ is a neighborhood of x in X if and only if $V(x)$ is a neighborhood of x in X_j for any j. A neighborhood here need not be an open set. The space X is not a topological space in general [6]. We will call X *the w-meet of* $\{X_j\}$.[5]

3.1. Theorem. *For any P_0^* space X, X is a w-meet of w-locally P spaces.*

Proof. The proof is almost similar to the one of Theorem 1.3.

Let X be a topological space and let $C(X)$ be the set of all continuous functions on X. Wada [7] has proved the following fact, which is useful for a kernel representation of compact linear oprators on $C(X)$ for a k_0 space X: If H ($\subset C(X)$, X is a k_0 space) is relative compact in the topology of $\mathfrak{S}$-convergence, then H is equicontinuous, where $\mathfrak{S}$ is the family of all compact subsets in X (as to the kernel representation, see R. E. Edwards: Functional analysis, theory, and applications (1965) p. 662).

We obtain moreover the following.

3.2. Theorem. *Let X be a P_0^* space. If H in $C(X)$ is precompact in the topology of $\mathfrak{S}$-convergence, then H is equicontinuous (and is pointwise bounded),*[6] *where $\mathfrak{S}$ is the family of all P-subsets in X.*

Proof. Since X is a P_0^* space, by Theorem 3.1, X is the w-meet of w-locally P spaces $\{X_K\}$. Then we can regard that $H \subset C(X_K)$ for any K and H is precompact in $C(X_K)$ under the topology of $\mathfrak{T}$-convergence, where $\mathfrak{T} = \{K$ and all points in $X_K \sim K\}$, so H is equicontinuous in X_K (cf. [1]), that is, for $\varepsilon > 0$, there is a neighborhood $U_K(x_0)$ such that for any $x \in U_K(x_0)$ and for any $u \in H$ $|u(x) - u(x_0)| < \varepsilon$. $W(x_0) = \bigcup U_K(x_0)$ is a neighborhood of X, so for any $x \in W(x_0)$ and for any $u \in H$ $|u(x) - u(x_0)| < \varepsilon$. This shows that H is equicontinuous in X.

In general, the converse of the theorem is false, that is, when X is a P_0^* space an equicontinuous set in $C(X)$ which is pointwise bounded need not be precompact in the topology of $\mathfrak{S}$-convergence. Suppose that P is "separable", then we see that any metric space

5) We used the symbol $\wedge_*$ or $\wedge$ in [6] in the place of $\wedge$ or w-meet.

6) H is pointwise bounded if any $x \in X$ $H(x) = \{f(x); f \in H\}$ is bounded.

is a P_0^* space. If X is a metric space and $\mathfrak{S}=\{$all separable subsets in $X\}$, any H having the hypothesis in Theorem 3.2 is equicontinuous, but we have the following:

Let X be a complete metric space and let $\mathfrak{S}_1$ be a family of subsets in X. Let the topology of $\mathfrak{S}_1$-convergence have the following property; any equicontinuous and pointwise bounded set H in $C(X)$ is always $\mathfrak{S}_1$-precompact. Then $\mathfrak{S}_1$ is a family of relative compact subsets in X.

For, since X is a complete metric space, we need only to prove that any S in $\mathfrak{S}_1$ is totally bounded (=precompact). Let $S_0(\in \mathfrak{S}_1)$ be not totally bounded, then S cannot be covered by a finite family of δ-spheres (for some $\delta>0$). We put $x_1 \in S_0$ arbitrarily. Then we can find a point $x_n \in S_0$ such that $\bigcup_{i=1}^{n-1} S_\delta(x_i) \not\ni x_n$ for any n, where $S_\delta(x_i)$ is the open δ-sphere such that the center is x_i. We put here $f_i(x)=\text{Max}\,\{\delta-d(x, x_i), 0\}$, where $d(x, y)$ is the distance function on X. Then we can prove that $H=\{f_i \mid i=1, 2, 3, \cdots\}$ is equicontinuous and pointwise bounded, so H is $\mathfrak{S}_1$-precompact by the hypothesis. But we see that $\|f_n-f_m\|_{S_0}=\sup_{x\in S_0} |f_n(x)-f_m(x)| \geqq \delta$ for $n\neq m$. This is a contradiction.

References

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