155. *Algebraic Proof of the Separation Theorem on Dummett's* LC

By Tsutomu HOSOI

Mathematical Institute, University of Tokyo

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Dummett's LC is an intermediate propositional calculus with the following axioms:

1.1 $p \supset q \supset p$.

1.2 $(p \supset q \supset r) \supset (p \supset q) \supset (p \supset r)$.

1.3 $((p \supset q) \supset r) \supset ((q \supset p) \supset r) \supset r$.

1.4 $p \,\&\, q \supset p$ and $p \,\&\, q \supset q$.

1.5 $(p \supset q) \supset (p \supset r) \supset (p \supset q \,\&\, r)$.

1.6 $p \supset p \vee q$ and $q \supset p \vee q$.

1.7 $(p \supset r) \supset (q \supset r) \supset (p \vee q \supset r)$.

1.8 $(p \supset \sim q) \supset (q \supset \sim p)$.

1.9 $\sim p \supset p \supset q$.

The rules of inference are modus ponens and the rule of substitution for variables. We associate to the right and assume the convention that $\supset$ binds less strongly than other connectives. This axiomatization is due to Bull [1].

The separation theorem is the following

Theorem. *A provable formula can be proved by using at most the axioms for implication and those of the connectives which actually appear in the formula.*

The separation theorem on this LC has been proved as a corollary in [4]. But here we show its algebraic proof. The algebraic proof of the theorem was first given by Horn [3] on the intuitionistic system. And we borrow some definitions and results from [3] without mentioning. (So see his paper for those.)

Horn's intuitionistic system is obtained from our system by deleting 1.3. So our definition of an I algebra must differ from his. Our I algebra must satisfy the following conditions:

2.1 If $1 \to x = 1$, then $x = 1$.

2.2 If $x \to y = y \to x = 1$, then $x = y$.

2.3 $x \to y \to x = 1$.

2.4 $(x \to y \to z) \to (x \to y) \to (x \to z) = 1$.

2.5 $((x \to y) \to z) \to ((y \to x) \to z) \to z = 1$.

The last is the one added to Horn's definition.

By Horn the proof of the theorem was reduced to the problem of embedding each algebra in an ICDN algebra.

3.1. **Lemma.** *Any* I (*or* IC, *or* ID, *or* ICD) *algebra can be embedded in an* IN (*or* ICN, *or* IDN, *or* ICDN) *algebra.*

Proof. This is proved just as the proof of Horn's theorem 5.1. We only need to add the case of 2.5.

(i) If $x=0$, $((x\to y)\to z)\to((y\to x)\to z)\to z=z\to((y\to 0)\to z)\to z=1$ by 2.3.

(ii) If $x\neq 0$ and $y=0$, $((x\to y)\to z)\to((y\to x)\to z)\to z$
$=((x\to 0)\to z)\to(z\to z)=1$.

(iii) If $x\neq 0$, $y\neq 0$, and $z=0$,
$((x\to y)\to z)\to((y\to x)\to z)\to z=0\to(0\to 0)=1$.

(iv) If x, y, and z are not 0, 2.5 holds obviously.

4.1. **Lemma.** *Any* IN (*or* IDN) *algebra can be embedded in an* ICN (*or* ICDN) *algebra.*

Proof. This is also proved just as Horn's 8.5. We add only the case of 2.5.

(i) If each of A, B, and C has only one member,
$((A\to B)\to C)\to((B\to A)\to C)\to C=1$ by 2.5.

(ii) If $A=a$, $B=b$, and $C=c\cup C'$, we only need to prove that
$((a\to b)\to c\cup C')\cup((b\to a)\to c\cup C')\to c=1$. But
$((a\to b)\to c\cup C')\cup((b\to a)\to c\cup C')\to c$
$=((a\to b)\to c)\cup((a\to b)\to C')\cup((b\to a)\to c)\cup((b\to a)\to C')\to c$
$\geqq((a\to b)\to c)\cup((b\to a)\to c)\to c=1$ by 2.5.

(iii) If $A=a$ and $B=b\cup B'$,
$((a\to b\cup B')\to C)\cup(b\cup B'\to a)\to C)\to C$
$=((a\to b)\cup(a\to B')\to C)\cup((b\cup B'\to a)\to C)\cup(b\cup B'\to a)\to C)\to C$
$=((a\to B')\to(a\to b)\to C)\cup((b\to B'\to a)\to C)\cup((B'\to b\to a)\to C)\to C$
[as $(((b\to a)\to B')\to C)\to(((B'\to b\to a)\to C)\to C)=1$ by the hypothesis]
$\geqq((a\to B')\to(a\to b)\to C)\cup((b\to B'\to a)\to C)\cup((b\to a)\to B')\to C$
[as $(b\to a)\to B'\leqq a\to B'$]
$\geqq((a\to B')\to(a\to b)\to C)\cup((b\to B'\to a)\to C)\cup(a\to B')\to C$
[by Horn's 7.3]
$\geqq((a\to b)\to C)\cup((b\to B'\to a)\to C)\to C$
[Here $B'\to a$ can be thought as one member. So by 2.5.]
$\geqq((a\to b)\to C)\cup((B'\to a)\to b)\to C$
[as $(B'\to a)\to b\leqq a\to b$]
$\geqq((a\to b)\to C)\cup(a\to b)\to C\geqq C\to C=1$.

(iv) If $A=a\cup A'$,
$((a\cup A'\to B)\to C)\cup((B\to a\cup A')\to C)\to C$
$=((B\to A')\cup(B\to a)\to C)\cup((a\to A'\to B)\to C)\cup((A'\to a\to B)\to C)\to C$
[as $((A'\to(a\to B))\to C)\to C\geqq((a\to B)\to A')\to C$ by the hypothesis]
$\geqq((B\to A')\cup(B\to a)\to C)\cup((a\to A'\to B)\to C)\cup((a\to B)\to A')\to C$

[as $(a \to B) \to A' \leqq B \to A'$]
$\geqq ((B \to A') \to (B \to a) \to C) \cup ((a \to A' \to B) \to C) \cup (B \to A') \to C$
[by Horn's 7.3]
$\geqq ((B \to a) \to C) \cup ((a \to A' \to B) \to C) \to C$
[as $((a \to A' \to B) \to C) \to C \geqq ((A' \to B) \to a) \to C$ by the hypothesis]
$\geqq ((B \to a) \to C) \cup ((A' \to B) \to a) \to C$
[as $(A' \to B) \to a \leqq B \to a$]
$\geqq ((B \to a) \to C) \cup (B \to a) \to C \geqq C \to C = 1.$

5.1. **Lemma.** *Any* ICN *algebra* $\mathcal{A}$ *can be embedded in an* ICDN *algebra* $\mathcal{B}$.

Proof. In $\mathcal{A}$ we define $a+b$ by $((a \to b) \to b)((b \to a) \to a)$. And we take $\mathcal{A}$ as $\mathcal{B}$ taking the identity mapping as the embedding mapping. In order to see that $\mathcal{B}$ is an ICDN algebra, we only need to show that the analogues of 1.6 and 1.7 are identically 1.

(i) $a \to a+b = a \to ((a \to b) \to b)((b \to a) \to a)$
$= (a \to ((a \to b) \to b)(a \to ((b \to a) \to a) = 1.$ And similarly $b \to a+b = 1.$

(ii) We put

$$A = (a \to c) \to (b \to c) \to (a+b \to c)$$
$$= (a \to c) \to (b \to c) \to ((a \to b) \to b) \to ((b \to a) \to a) \to c.$$

Then by 2.5, $((a \to b) \to A) \to ((b \to a) \to A) \to A = 1$. On the other hand,

$$(a \to b) \to A = (a \to b) \to (a \to c) \to (b \to c) \to ((a \to b) \to b) \to ((b \to a) \to a) \to c$$
$$\geqq (a \to b) \to ((a \to b) \to b) \to (b \to c) \to c = 1,$$

and likewise $(b \to a) \to A = 1$. Hence $A = 1$.

By the above three lemmas, the embedding problem has been solved.

References

[1] R. A. Bull: The implicational fragment of Dummett's LC. J. Symbolic Logic, **27**, 189-194 (1962).
[2] M. Dummett: A propositional calculus with denumerable matrix. J. Symbolic Logic, **24**, 96-107 (1959).
[3] A. Horn: The separation theorem of intuitionist propositional calculus. J. Symbolic Logic, **27**, 391-399 (1962).
[4] T. Hosoi: On the separation theorem of intermediate propositional calculi. Proc. Jap. Acad., **42**, 535-538 (1966).