153. *Another Proof of Two Decomposition Theorems of Semigroups*

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1. **Introduction.** One of the early decomposition theorems for semigroups was given by David McLean [2] and may be stated as follows:

Theorem. *An idempotent semigroup S has a greatest semilattice decomposition into rectangular bands.*

In his proof McLean defines a relation σ on S by

$$a\sigma b \text{ if and only if } aba=a \text{ and } bab=b$$

σ is then shown to be the smallest semilattice congruence (abbr. s-congruence) on S. That is, S/σ is a semilattice, and if S/σ' is a semilattice then $\sigma\subseteq\sigma'$. The most difficult part of this proof is in showing the transitivity of σ. We will give another proof based on the concept of "content" of a semigroup and a theorem of T. Tamura [4]. Finally we will give another proof of the following theorem of T. Tamura and N. Kimura [3].

Theorem. *A commutative semigroup S has a greatest semilattice decomposition into archimedean semigroups.*

2. **Contents.** **Definition 1.** Let $a_1, a_2, \cdots, a_n$ be elements of a semigroup S. The "content" of $a_1, a_2, \cdots, a_n$ in S, $C_S\langle a_1, a_2, \cdots, a_n\rangle$, is the set of elements of S which can be expressed as a product involving all the elements $a_1, a_2, \cdots, a_n$.

From the definition it is obvious that $C_S\langle a_1, a_2, \cdots, a_n\rangle$ is a subsemigroup of S. As a special case we consider a band.

Lemma 1. *Let S be a band. Then any content $C_S\langle x_1, x_2, \cdots, x_n\rangle$ is a rectangular band.*

To prove Lemma 1 it is sufficient to prove Lemma 2.

Lemma 2. *Let F be a free band generated by $a_1, a_2, \cdots, a_n$. A content $C_F\langle a_1, a_2, \cdots, a_n\rangle$ is a rectangular band.*

However we will prove Lemma 4 which is a more generalized form of Lemma 2.

Let F be the free band generated by $G=\{g_\lambda : \lambda\in\Lambda\}$.

Definition 2. If $X\in F$, let $G(X)=\{g_\lambda\in G\colon X=g_{\lambda 1}g_{\lambda 2}\cdots g_{\lambda n}\}$.[1]

Lemma 3. *If $X, Y\in F$ then*

(i) $G(XY)=G(X)\cup G(Y)$

1) A similar definition was used by Green and Rees [1].

(ii) $G(XY)=G(YX)$.

The proof is a trivial result of Definition 2.

Lemma 4. *If $X, Y \in F$, then $G(Y) \subseteq G(X) => XYX=X$.*

Proof. Suppose $X \in F$, $a \in G$, and $X=UaV$ where $U, V \in F$ may be empty for convenience of proof. Then

(1) $$X=UaV=U(aV)(aV)=XaV.$$

The proof of this Lemma is by induction on the length of Y.

If the length of Y is 1, then $G(Y)=\{a\}$ and $a \in G$. Hence $X=UaV$ and using (1)

$$XYX=XaX=Xa(XaV)=XaV=X.$$

Now assume the lemma holds for all X, Z with the length of Z less than or equal to n where $G(Z) \subseteq G(X)$. Suppose Y has length $n+1$ and $G(Y) \subseteq G(X)$. $Y=aZ$ where $a \in G$ and $Z \in F$. By Lemma 3

$$G(Y)=G(aZ)=G(Z) \cup \{a\} \subseteq G(X)$$

so we may apply (1)

(2) $$XYX=X(aZ)X=XaZXaV=(Xa)Z(Xa)V.$$

Now $G(Z) \subseteq G(X)=G(Xa)$ and the length of Z is n, so by the induction assumption $(Xa)Z(Xa)=Xa$ which combined with (1) and (2) gives

$$XYX=(Xa)Z(Xa)V=XaV=X.$$

Thus we have proved Lemma 4 and hence Lemma 2. Since $C_S\langle x_1, x_2, \cdots, x_n\rangle$ is a homomorphic image of $C_F\langle x_1, \cdots, x_n\rangle$, we obtain Lemma 1.

Definition 3. Let S be a semigroup and define relations ρ_1 and ρ on S by $a\rho_1 b$ if and only if a and b are in a content $C_S\langle x_1, x_2, \cdots, x_n\rangle$ for some $x_1, x_2, \cdots, x_n$. Let ρ be the transitive closure of ρ_1, that is $a\rho b$ if and only if there are $a_1, a_2, \cdots, a_n \in S$ such that $a=a_1$, $b=a_n$, and $a_i\rho_1 a_{i+1} (i=1, \cdots, n-1)$[5].

Lemma 5. *ρ is the smallest s-congruence on S.*

Proof. It is easy to see that ρ is an s-congruence. We have to prove that ρ is smallest. Let ρ' be an s-congruence on S. Suppose $a\rho b$. Then $a=a_0, a_1, \cdots, a_n=b$ such that a_i and a_{i+1} are in $C_S\langle x_1, \cdots, x_{k_i}\rangle$. We can easily prove that if a_i and a_{i+1} are in $C_S\langle x_1, x_2, \cdots, x_{k_i}\rangle$ then $a_i\rho' a_{i+1}$. Accordingly $a\rho b$ implies $a\rho' b$.

Lemma 6. *If S is a band, then $\rho=\rho_1$. That is, ρ_1 is the smallest s-congruence on S.*

Proof. We know ρ_1 is reflexive, symmetric and compatible. To prove transitivity, suppose $a\rho_1 b$ and $b\rho_1 c$. By Lemma 1

$$a=aba;\ bab=b;\ c=cbc;\ bcb=b.$$

Hence $a=aba=a(bcb)a$ and $c=cbc=c(bab)c$ so $a, c \in C_S\langle a, b, c\rangle$. Therefore $a\rho_1 c$.

3. Bands and Commutative Semigroups. Let σ be a relation defined in § 1.

$a\sigma b$ if and only if $aba=a$ and $bab=b$.

Theorem 1. *Let S be a band. Then $\rho_1=\sigma$. In other words σ is the smallest s-congruence on S.*

Proof. If $a\sigma b$, then a and b are in $C_S\langle a, b\rangle$, namely $a\rho_1 b$. Hence $\sigma\subseteq\rho_1$. Next assume $a\rho_1 b$, that is, a and b are in a content of S. By Lemma 1, $aba=a$ and $bab=b$ so $a\sigma b$. Therefore $\rho_1=\sigma$. By Lemmas 5 and 6, σ is the smallest s-congruence.

Theorem 2. *Let S be a commutative semigroup. Define a relation τ on S by*

$a\tau b$ if and only if $a^m=bx, b^n=ay$ for some $m>0, n>0, x, y\in S$. Then τ is the smallest s-congruence on S.

Proof. Lemma 5 may be used to prove Theorem 2 as follows. First prove that τ is transitive, and prove that $\rho_1\subseteq\tau$, so $\rho\subseteq\tau$. We can prove $\tau\subseteq\rho$ since

$$a\rho a^m=bx\rho b^{n+1}x=abxy=a^{m+1}y\rho ay=b^n\rho b.$$

Thus we have $\rho=\tau$.

References

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