151. On Positive Irreducible Operators in an Arbitrary Banach Lattice and a Problem of H. H. Schaefer

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The purpose of this paper is to report our recent result concerning the spectrum of a positive operator in a Banach lattice, namely the following general

Theorem. *Let E be a Banach lattice and $T\in\mathfrak{L}(E)$ satisfy the following three conditions:*

I) *T is positive.*

II) *T is irreducible, i.e., there exists no non-trivial closed solid*[1] *subspace of E which is invariant under T.*

III) *The value $\lambda=r(T)$*[2] *is a pole of the resolvent $R(\lambda, T)$. Then on the spectral circle $\Gamma=\{\lambda; |\lambda|=r(T)\}$, the spectrum of T coincides with the set of k-th roots of unity multiplied by $r(T)$, each of which is a simple pole of $R(\lambda, T)$, where k is a fixed positive integer determined by T.*

In the previous papers [5], [6], and [8] the present authors have proved the corresponding theorems in the special cases where E is the $l_p(1<p<\infty)$, $L_p(1<p<\infty)$, and $C(S)$. As a continuation of these investigations we can prove this theorem in the general case where E is an arbitrary Banach lattice. But here we will sketch the outline of the proof, the full detail of which will be published elsewhere.

Before entering the proof, we give some preliminary remarks and propositions. Let E be a Banach lattice and K be the positive cone of E, the duals of which will be denoted by E^* and K^* respectively. An element $x\in K$ is called quasi-inner if the order interval $[0, x]$ is total in E, and non-support if it is not a support point of the convex set K. A positive operator $T\in\mathfrak{L}(E)$ is called semi-non-support if, for each non-zero $x\in K$ and for each non-zero $f\in K^*$, there exists a positive integer n such that

$$f(T^n x)>0.$$

For any $e\in K$, an element $x\in E$ is called bounded w.r.t. e if there exists a positive number c which satisfies

$$|x|\leqq ce.$$

1) A subset F of E is solid if $|x|\leqq|y|$ and $y\in F$ imply $x\in F$.
2) $r(T)$ is the spectral radius of T, i.e., $r(T)=\mathrm{Max}\{|\lambda|; \lambda\in\sigma(T)\}$.

The set of elements x bounded w.r.t. e will be denoted by E_e. For $x \in E_e$, define a new norm

$$\| x \|_e = \inf \{c; |x| \leqq ce\}.$$

Then, as was shown in H. H. Schaefer [10], E_e becomes an AM space. By Kakutani [3], E_e is represented by $C(\mathfrak{M})$, i.e., the space of continuous functions on a compact Hausdorff space $\mathfrak{M}$. Using this representation, we can prove

Proposition 1. *For any $x, y \in E$*

$$\bigvee_{0 \leqq \theta \leqq 2\pi} ((\cos\theta)x + (\sin\theta)y)$$

exists in E.

As a consequence of this proposition, every Banach lattice (even if it is not σ-complete) can be camplexificated with extended absolute value.[3] We can also prove

Proposition 2. *Let e be an element of K. Then e is non-support if and only if for any $x \in K$, $x \wedge ne$ converges to x strongly in E.*

Proposition 3. *An element $x \in K$ is quasi-inner if and only if it is non-support.*

Proposition 4. *A positive operator $T \in \mathfrak{L}(E)$ is irreducible if and only if it is semi-non-support.*

By theorem 2 in [7] we have

Proposition 5. *Let $T \in \mathfrak{L}(E)$ satisfy conditions* I) *and* III) *in the theorem. Then T is irreducible if and only if the following three conditions* 1), 2), *and* 3) *are satisfied.*

1) *The eigenspace of T for $r(T)$ is one-dimensional.*

2) *The eigenspace of T for $r(T)$ contains a quasi-inner element.*

3) *The eigenspace of T^* for $r(T)$ contains a strictly positive functional.*[4]

We have further

Proposition 6. *Let f_n be a non-decreasing sequence of K^* and $f_0 \in K^*$. Then f_n w^*-converges to f_0 if and only if*

$$\bigvee_n f_n = f_0.$$

Hereafter we assume, for the sake of simplicity, that $r(T)=1$. We also assume that e and f_0 are the eigen elements for 1 of T and T^* respectively. Moreover we normalize these elements by

$$\| e \| = 1 \text{ and } f_0(e) = 1.$$

For the proof of the theorem we have to show the following three propositions, namely,

A) *$P_\sigma(T)$[5] $\cap \Gamma$ coincides with the set of k-th roots of unity for*

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3) For this, see H. H. Schaefer [10], p. 274.

4) An element $f \in K^*$ is strictly positive if $x \in K$ and $x \neq 0$ imply $f(x) > 0$.

a positive integer k, each element of which is a simple pole of $R(\lambda, T)$.

B) $R_\sigma(T) \cap \Gamma = \phi$.

C) $C_\sigma(T) \cap \Gamma = \phi$.

As for A), it is readily seen that we have only to consider the space E_e, the problem being reduced to the case of $C(\mathfrak{M})$. Although the restriction of T to E_e may not fulfil II) or III), the discussions of theorem 3.3 in H. H. Schaefer [10] remain true in this case. Then, applying propositions 2 and 3, we can prove A).

As for B), we follow the corresponding part in [8]. In the first place we difine the projection operator $P_0 \in \mathfrak{L}(E^*)$ by

$$P_0 f = \bigvee_n (f \wedge n f_0) \qquad (f \in K^*),$$

and extend this to E^*. We also consider $E^*_{f_0}$, i.e., the set of $f \in E^*$ which is bounded w.r.t. f_0, then clearly

$$E^*_{f_0} \subset P_0 E^* \subset E^*.$$

It can be easily seen that these spaces are invariant under T^*. The restrictions of T^* to $E^*_{f_0}$ and $P_0 E^*$ will be denoted by T^*_0 and T^*_1 respectively. By definitions we see

$$R_\sigma(T) \cap \Gamma \subset P_\sigma(T^*) \cap \Gamma. \tag{1}$$

and

$$P_\sigma(T^*) \cap \Gamma \subset P_\sigma(T^*_0) \cap \Gamma. \tag{2}$$

Let

$$\lambda_0 \in P_\sigma(T^*_0) \cap \Gamma.$$

Then, as in the proof of A) in the preceding part, there exists a bijective operator $D \in \mathfrak{L}(E^*_{f_0})$ satisfying

$$T^*_0 = \lambda_0 D^{-1} T^*_0 D \text{ and } |Df| = |D^{-1} f| = |f| \qquad (f \in E^*_{f_0}).$$

By applying proposition 6 to the operator D, we can extend D to a bijective operator of $\mathfrak{L}(P_0 E^*)$ which will be denoted by the same letter D. Then

$$|Df| = |D^{-1} f| = |f| \qquad (f \in P_0 E^*).$$

Since T^* is w^*-continuous, we also have

$$T^*_1 = \lambda_0 D^{-1} T^*_1 D.$$

Then, by the reduction theory in [8] which remains true in this case, we can show that λ_0 is a pole of $R(\lambda, T^*)$ and consequently

$$\lambda_0 \in P_\sigma(T).$$

Therefore

$$P_\sigma(T^*_0) \cap \Gamma \subset P_\sigma(T) \cap \Gamma. \tag{3}$$

The relations (1), (2), and (3) completes the proof of this part.

As for C), we introduce a new norm $\|x\|_L = f_0(|x|)$, and consider the norm completion L of E w.r.t. this norm. Then it easy to see

5) $P_\sigma(T)$, $R_\sigma(T)$ and $C_\sigma(T)$ are the point spectrum, residual spectrum and continuous spectrum respectively.

that L is an AL space. By Kakutani [2], L can be represented by a concrete L_1. We can extend T to L, which we denote by T_L. T_L may not satisfy II) or III) in the theorem. However, instead of III), we have

Proposition 7. *Let $x_n \in L$ satisfy the following conditions:*

$$|x_n| \leqq e$$

and

$$\| T_L x_n - x_n \|_L \to 0.$$

Then we have

$$\| x_n - f_0(x_n) e \|_L \to 0.$$

Using this proposition we can proceed as in [6] and get

$$C_\sigma(T) \cap \Gamma \subset P_\sigma(T_L) \cap \Gamma. \tag{4}$$

We then apply theorem 5 in S. Karlin [4] and theorem 3.9 in N. Dunford [1] and get

$$P_\sigma(T_L) \cap \Gamma \subset P_\sigma(T) \cap \Gamma. \tag{5}$$

The relations (4) and (5) completes the proof of this part.

It seems to be interesting that in discussing A) and B) we restrict the operators T and T^* to AM spaces E_e and $E^*_{f_0}$ respectively, while in discussing C) we extend T to an AL space L.

H. H. Schaefer proposed in [9] the following two problems concerning a positive operator T:

a) *If $r(T)$ is an isolated singularity of $R(\lambda, T)$, is every singularity of $R(\lambda, T)$ on Γ isolated?*

b) *If $r(T)$ is a pole of $R(\lambda, T)$, does there exist on Γ no singularity other than poles?*

As a consequence of our theorem, problem b) is answered affirmatively for irreducible operators in an arbitrary Banach lattice.

Finally we will make a few remarks concerning the case where the operator T, which satisfy conditions I) and III) of our theorem, is not necessarily irreducible. Proposition 5 states that condition II) is equivalent to conditions 1), 2), and 3). If we replace condition 1) by a weaker one, namely,

1′) the eigenspace of T for $r(T)$ is finite-dimensional,

then the conclusions of the theorem remain true with only a slight modification. But if condition 1) or 1′) is not assumed at all, then, with all other conditions satisfied, the conclusions of the theorem fail to hold. This can be shown by a simple example in the concrete space $l_p (1 \leqq p \leqq \infty)$. This example also provides a negative answer to either of the above problems a) and b) for T which is not necessarily irreducible.

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