194. On Axiom Systems of Propositional Calculi. XXIV

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In his well known book [2], A. Church defines a propositional calculus P_1 which is equivalent to the classical propositional calculus. The calculus with a propositional constant 0 (which corresponds to the false proposition) is given by the following axioms:

1 $CpCqp$,
2 $CCpCqrCCpqCpr$,
3 $CCCp00p$.

In [1], by two inference rules of substitution and detachment we proved that the first two axioms imply the following theses:

4 Cpp,
5 $CCpqCCqrCpr$,
6 $CCqrCCpqCpr$,
7 $CCpCqrCqCpr$,
8 $CCpCpqCpq$.

In this note, using substitution and detachment rules, we shall show that the calculus is the classical one. To do so, put $Np=Cp0$, then the axiom 3 means $CNNpp$. We deduce an axiom system by J. Lukasiewicz (see [3]):

a $CCpqCCqrCpr$,
b $CCNppp$,
c $CpCNpq$.

The thesis a follows from the axioms 1 and 2, as already well-known so we must prove that the theses b and c hold in the P_1-calculus. For the proofs, we use the proofline method.

1 $p/CCCp00p$, $q/0$ $*C3—9$,
9 $C0CCCp00p$.
2 $p/0$, $q/CCp00$, r/p $*C9—C1$ $p/0$, $q/Cp0—10$,
10 $C0p$.
1 $p/C0q$, q/p $*C10$ $p/q—11$,
11 $CpC0q$.
2 $q/0$, r/q $*C10—12$,
12 $CCp0Cpq$.
6 $p/Cp0$, q/p, $r/q*C12—13$,
13 $CpCCp0q$.

This means $CpCNpq$, which is the thesis c.

7 $p/Cp0$, q/p, $r/0$ $*C4$ $p/Cp0—14$,

14 $CpCCp00$.

5 $p/CCp0CCp00$, $q/CCp00$, r/p $*C8$ $p/Cp0$, $q/0$—$C3$—15,

15 $CCCp0CCp00p$.

6 $p/Cp0$, q/p, $r/CCp00$ $*C14$—16,

16 $CCCp0pCCp0CCp00$.

6 $p/CCp0p$, $q/CCp0CCp00$, r/p $*C15$—$C16$—$C17$,

17 $CCCp0pp$,

which means $CCNppp$. Therefore we complete the proof.

References

[1] Y. Arai and K. Iséki: On axiom systems of propositional calculus. VII. Proc. Japan Acad., **41**, 667-669 (1965).
[2] A. Church: Introduction to Mathematical Logic. Princeton (1956).
[3] J. Lukasiewicz: Elements of Mathematical Logic. Oxford (1963).