188. *A Series of Successive Modifications of Peirce's Rule*

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After Ono [2], we denote by **LOS** the sentence-logical part of the *primitive logic* **LO** [1]. **LOS** is the logic having → (*implication*) as the only logical constant. We may axiomatize **LOS** as follows:

(1) $$p\to(q\to p),$$

(2) $$(p\to(q\to r))\to((p\to q)\to(p\to r)),$$

with substitution and detachment (*modus ponens*) as the only rules of inference. (p, q, r are three distinct proposition-variables.) Next, we denote by **LOQS** a logic obtained from **LOS** by adding

(3) $$((p\to q)\to p)\to p, \quad (\textit{Peirce's rule } [3]),$$

to the axioms of **LOS**. We can easily see that Peirce's rule is not provable in **LOS**. Hence, **LOS** is weaker than **LOQS**. (Notation: **LOS**⊂**LOQS**.)

On the advice of Prof. K. Ono, we studied the following problem: "Does there exist a logic **L** such that **LOS**⊂**L**⊂**LOQS**?" This problem has been solved in the affirmative. Namely, we have recognized the fact that we can obtain a series of successive modifications of Peirce's rule, by substituting the foregoing modified Peirce's rule in place of q in the proposition (3) (Peirce's rule) over and over again renewing p each time. The purpose of the present paper is to introduce a method for weakening Peirce's rule and to give a series of successive modifications of Peirce's rule. The author would wish to express his thanks to Prof. K. Ono for his kind guidance and encouragement.

§ 1. To begin with, we explain a first step of the above-mentioned method. In order to prove that the proposition (3) is not provable in **LOS**, we usually make use of the matrix[1] $N=\langle\{0, 1, 2\}, \{0\}, \to_N\rangle$, where

$$a\to_N b=\begin{cases} b \text{ if } a<b, \\ 0 \text{ otherwise.}\end{cases}$$

Namely, the propositions (1) and (2) are satisfied by N, but (3) is not satisfied by N. (Here, a proposition P is said to be satisfied by N if and only if P takes the value 0 identically with respect to N.) In fact, we can easily see the following:

1) As for matrices, see Rose [4] for example.

$$((a \to_N b) \to_N a) \to_N a = \begin{cases} 1 \text{ if } a=1 \text{ and } b=2, \\ 0 \text{ otherwise.} \end{cases}$$

Hence, (3) receives the value 1 if we assign the value 1 to p, the value 2 to q. Moreover, it is remarkable that Peirce's rule never takes the value 2 and that it is necessary to assign the value 2 to q in order that (3) should receive the value 1.

Now, let us consider the following proposition (3*) obtained from (3) (Peirce's rule) by substituting a proposition, $((p^* \to q) \to p^*) \to p^*$ (also Peirce's rule), for q in (3):

(3*) $$((p \to [((p^* \to q) \to p^*) \to p^*]) \to p) \to p.$$

(p, p*, q are three distinct proposition-variables.) From the remark above described, we can conclude that (3*) is satisfied by N. Furthermore, we can prove that (3*) is not provable in **LOS** by the use of the matrix $N^* = \langle \{0, 1, 2, 3\}, \{0\}, \to_{N^*} \rangle$, where

$$a \to_{N^*} b = \begin{cases} b \text{ if } a<b, \\ 0 \text{ otherwise.} \end{cases}$$

(It is easy to check that (1) and (2) are satisfied by N^*, whereas (3*) is not satisfied by N^*.) Therefore, we can assert that (3*) is a really restricted rule of Peirce's rule.

§2. By extending the method described in §1, we can weaken Peirce's rule successively. Let us consider a series of propositions $P_1, P_2, \cdots$ defined recursively as follows:

$$\begin{cases} P_1 \equiv ((p \to q) \to p) \to p, \\ P_{n+1} \equiv ((p_n \to P_n) \to p_n) \to p_n, \quad (n=1, 2, \cdots), \end{cases}$$

where p, q, p_n's are *mutually distinct*[2] proposition-variables. We denote by **LOS**$[P_n]$ a new logic obtained from **LOS** by adding P_n to the axioms of **LOS**. P_1 is Peirce's rule, so **LOS**$[P_1]$ coincides with **LOQS**. (Notation: **LOS**$[P_1]$ = **LOQS**.) It is clear that, for any $n \geq 1$, P_{n+1} is provable in **LOS**$[P_n]$.

Now, we would like to show that, for any $n \geq 1$, P_n is not provable in **LOS**$[P_{n+1}]$. For this purpose, we use the matrices $M_n = \langle \{0, 1, \cdots, n\}, \{0\}, \to_{M_n} \rangle$, where $n=1, 2, \cdots$, and

$$a \to_{M_n} b = \begin{cases} b \text{ if } a<b, \\ 0 \text{ otherwise.} \end{cases}$$

(M_1 corresponds to the ordinary two-valued truth-table, and M_2, M_3 accord with N, N^*, respectively.)

It is easy to check the following.

Lemma 1. *For any* $n \geq 1$, (1) *and* (2) (*axioms of* **LOS**) *are satisfied by* M_n.

The following lemma is also proved easily by mathematical

2) This restriction is really necessary since P_n turns out to be provable in **LOS** unless the variables p, q, p_n's are mutually distinct.

induction.

Lemma 2. *For any* $n \geq 1$, P_n *is satisfied by* M_n.

From these lemmas and the fact that, for any $n \geq 1$, a proposition Q is satisfied by M_n whenever two propositions P and $P \rightarrow Q$ are satisfied by M_n, we have the following.

Lemma 3. *For any* $n \geq 1$, *every provable proposition in* **LOS**$[P_n]$ *is satisfied by* M_n.

As stated in §1, P_1 is not satisfied by M_2, and P_2 is not satisfied by M_3. More generally, the following holds.

Lemma 4. *For any* $n \geq 1$, P_n *is not satisfied by* M_{n+1}.

From Lemmas 3 and 4, we have the following.

Lemma 5. *For any* $n \geq 1$, P_n *is not provable in* **LOS**$[P_{n+1}]$.

By virtue of Lemma 5 and the fact that, for any $n \geq 1$, P_{n+1} is provable in **LOS**$[P_n]$, we can establish the following theorem.

Theorem. $\mathbf{LOS} \subset \cdots \subset \mathbf{LOS}[P_n] \subset \cdots \subset \mathbf{LOS}[P_2] \subset \mathbf{LOS}[P_1] = \mathbf{LOQS}$.

References

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