254. Boolean Multiplicative Closures. II

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In this paper, we shall continue our discussion on Boolean multiplicative closures. The object of this paper is to prove main theorems by using the results of § 2.

3. Boolean multiplicative closures. We recall that the elements $x, y \in L$ are said to be *orthogonal* if $x \wedge y = 0$.

3.1. Lemma. *If* ∇ *fulfills conditions* C0), C1), *and* C5), *and if* x, y *are orthogonal elements of* L *such that* $x \wedge y = k \in I(\nabla)$, *then* $x \in I(\nabla)$ *and* $y \in I(\nabla)$.

Proof. By C0), C5), and the orthogonality of x and y we have:

(1) $$0 = \nabla(x \wedge y) = \nabla x \wedge \nabla y.$$

From (1) and C1) we have:

(2) $$y \wedge \nabla x \leq \nabla y \wedge \nabla x = 0.$$

Furthermore, as $x \leq x \vee y = k \in I(\nabla)$, and recalling that C5) implies C3), we have:

(3) $$\nabla x \leq \nabla k = k.$$

Using (2), (3), C3) and the fact that L is distributive, we get:

$$\nabla x = \nabla x \wedge k = \nabla x \wedge (x \vee y) = (\nabla x \wedge x) \vee (\nabla x \wedge y) = x \vee 0 = x,$$

i.e., $x \in I(\nabla)$. Interchanging x and y we have $y \in I(\nabla)$. Q.E.D.

Using the non-distributive lattice with five elements shown in ([2], figure 1, d, page 6) we can see that the distributive condition on L may not be omitted, in general, from 3.1.

We denote by $B = B(L)$ the Boolean algebra of all complemented elements of L. If $b \in B$, $-b$ denotes the complement of b.

An immediate consequence of 3.1 is:

3.2. Theorem. *Let* ∇ *be as in* 3.1. *Then* $B \subset I(\nabla)$.

A *Boolean multiplicative closure operator* ∇ defined on L is an operator ∇ defined on L such that $\nabla \in \mathrm{Com}(L)$ and $I(\nabla) \subset B(L)$.

We are going to characterize the class $\mathcal{L}$ of all distributive lattices with zero and unit that admits a Boolean multiplicative closure operator.

First of all, we note that according to 3.2., the conditions $\nabla \in \mathrm{Com}(L)$ and $I(\nabla) \subset B(L)$ imply that $I(\nabla) = \nabla(L) = B(L)$. So, if there exists a Boolean multiplicative closure operator ∇ on L it is unique, and moreover, as $B(L)$ is a sublattice of L, $\nabla \in \mathrm{Coam}(L)$ (see 1.1.). Therefore, to solve our problem we must reformulate the results of § 2 for the case $K=I(\nabla)=B$.

With the aid of the well known theorem on the equivalency between the notions of prime and maximal ideals in Boolean algebras, together with the results on S-prime and S-maximal ideals at the beginning of § 2, we can prove:

3.3. Lemma. *An ideal I of L is B-prime if and only if it is B-maximal ($B=B(L)$).*

From the above remarks, 2.6. and 3.3. it follows that:

3.4. Theorem. *$L\in \pounds$ if and only if $B=B(L)$ is lower relatively complete in L and every B-maximal ideal is a prime ideal of L. In this case, the Boolean multiplicative closure operator ∇ defined on L is unique, and $\nabla(L)=B$.*

We are going to give an intrinsic characterization of B-maximal filters.

3.5. Lemma. *If a B-maximal ideal M is a prime ideal of L, then M is a minimal prime ideal.*

Proof. Let F denote the complementary set of M (with respect to L). As M is a prime ideal of L, F is a filter. We shall prove that F is a maximal filter. Assume that $\bar{F}$ is a filter of L such that $F\subset\bar{F}$ and $F\neq\bar{F}$. Hence, there exists an element x satisfying:

(1) $x\in\bar{F}$ and (2) $x\notin F$.

But (2) is equivalent to $x\in M$ and as M is a B-ideal, there exists an element b such that:

(3) $b\in M_1=M\cap B$ and (4) $x\leq b$.

From (3) we have:

(5) $-b\notin M_1$ and then (6) $-b\in F\subset\bar{F}$.

By (4) we have:

(7) $$x\wedge -b\leq b\wedge -b=0$$

and from (1), (6), and (7) we get that $0\in\bar{F}$, i.e., $\bar{F}=L$, and we have proved that L is a maximal filter, so M is a minimal prime ideal.

Q.E.D.

2.9 together with 3.5 provides us a proof of:

3.6. Theorem. *If $L\in\pounds$, then M is a B-maximal ideal if and only if M is a minimal prime ideal of L.*

4. Remarks on multiplicative closures. Let S be an inf-semilattice (i.e. S is a partially ordered set that for any pair $x,y\in S$ there exists $x\wedge y\in S$), with unit 1. Obviously, we can define the class $\mathrm{Cm}(S)$ as in the case of lattices, and the inf-semilattices are the most general structure that admits such definition.

An element $i\in S$, $i\neq 1$, is called *subirreducible* if for any pair $x,y\in S$, $x\wedge y\leq i$ implies that $x\leq i$ or $y\leq i$. We shall say that S has the *subdecomposition property* in case every element of S different from 1 is a meet of subirreducible elements.

If K is a subsemilattice of S, an element $k \in K$ is called K-subirreducible if it is subirreducible in the semilattice K. We shall say that K is *subcompatible* if every K-subirreducible element is subirreducible in S, and we shall denote by $\mathrm{Rc}(S)$ the set of lower relatively complete and subcompatible subsemilattices of S.

It is easy to prove the following:

4.1. Theorem. *If* $\nabla \in \mathrm{Cm}(S)$, *then* $I(\nabla) \in \mathrm{Rc}(S)$.

We can construct examples that show that the condition $I(\nabla) \subset \mathrm{Rc}(S)$ is not sufficient in order that $\nabla \in \mathrm{Cm}(S)$. Nevertheless, with arguments similars to those used in the proof of 2.6., we can prove:

4.2. Theorem. *If* $K \in \mathrm{Rc}(S)$ *and* K *has the subdecomposition property, then the operator* ∇ *defined by* (1) *of* 1.1 *belongs to* $\mathrm{Cm}(S)$.

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