252. An Algebraic Formulation of K-N Propositional Calculus

By Kiyoshi Iséki

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A K-N axiom system of propositional calculus is given by J. B. Rosser (2). His axiom system of classical propositional calculus is written in the form of

a) $CpKpp$,

b) $CKpqp$,

c) $CCpqCNKqrNKrp$,

where functors K, N, C denote conjunction, negation, and implication respectively.

As well known, we have $Cpq=NKpNq$. Therefore Rosser's axiom system is denoted by two functors K, N as follows:

a′) $NKpNKpp$,

b′) $NKKpqNp$,

c′) $NKNKpNqNNKNKqrNNKrp$.

On the other hand, B. Sobociński obtained two new axiom systems which is equivalent to Rosser's system (see B. Sobociński [3], [4]). C. A. Meredith gave an axiom system (see C. A. Meredith and A. N. Prior [1]).

In the K-N propositional calculus, there are two rules of procedure:

1) One of them is the rule of substitution commonly used in the propositional calculus.

2) The other is the rule of detachment as follows. If $NK\alpha N\beta$ and α are theses, then β is also a thesis.

From Rosser's system or KN-system, we can define an algebraic system as follows: Let x be an abstract algebra consisting of $0, p, q, r, \cdots$ with a binary operation $*$ and a unary operation $\sim$ satisfying the following conditions:

1) $\sim(p*p)*p=0$,

2) $\sim p*(q*p)=0$,

3) $\sim\sim(\sim\sim(p*r)*\sim(r*q))*\sim(\sim q*p)=0$,

4) Let α, β be expressions in X, then $\sim\sim\beta*\sim\alpha=0$ and $\alpha=0$ imply $\beta=0$.

Then X is called *KN-algebra*. The condition 4) corresponds to the rule of detachment.

First of all, we shall prove some general theorems. The Greek letters denote expressions in X.

A) $\sim\alpha*\beta=0$ *implies* $\sim\sim(\beta*\gamma)*\sim(\gamma*\alpha)=0$.

Proof. In 3), put $p=\beta$, $q=\alpha$, $r=\gamma$, then $\sim\sim(\sim\sim(\beta*\gamma)*\sim(\gamma*\alpha))*\sim(\sim\alpha*\beta)=0$. By 4), we have $\sim\sim(\beta*\gamma)*\sim(\gamma*\alpha)=0$.

Then we have the following

B) $\sim\alpha*\beta=0, \gamma*\alpha=0$ *imply* $\beta*\gamma=0$.

C) $\sim\alpha*\beta=0$, $\sim\gamma*\alpha=0$ *imply* $\beta*\sim\gamma=0$.

In A), put $\alpha=(p*p)$, $\beta=p$, $\gamma=\sim p$, then $\sim(p*p)*p=0$ implies

$$\sim\sim(p*\sim p)*\sim(\sim p*(p*p))=0.$$

By 2), we have

5) $p*\sim p=0$.

In 3), put $p=\sim\sim q$, $r=\sim r$, then

$$\sim\sim(\sim\sim(\sim\sim q*\sim r)*(\sim(\sim r*q))*\sim(\sim q*\sim\sim q)=0.$$

By 5), $\sim q*\sim\sim q=0$, hence

6) $\sim\sim(\sim\sim q*\sim r)*\sim(\sim r*q)=0$.

In 3), put $p=\sim\sim q$, then by 5)

$$\sim\sim(\sim\sim q*r)*\sim(r*q)=0.$$

This expression implies

D) *If* $\alpha*\beta=0$, *then* $\sim\sim\beta*\alpha=0$, *and* $\sim\sim\alpha*\sim\sim\beta=0$.

5) and 6) imply

7) $\sim\sim\sim p*p=0$.

Let $\sim\sim\alpha*\sim\beta=0$, put $p=\sim\beta$, $q=\sim\alpha$, $r=\alpha$ in 3), then

$$\sim\sim(\sim\beta*\alpha)*\sim(\alpha*\sim\alpha)=0,$$

and we have $\sim\beta*\alpha=0$. Hence

E) $\sim\sim\alpha*\sim\beta=0$ *implies* $\sim\beta*\alpha=0$.

Let $\sim\beta*\alpha=0$, put $p=\alpha$, $q=\beta$, $r=\gamma$ in 3), then we have

$$\sim\sim(\alpha*\gamma)*\sim(\gamma*\beta)=0.$$

By E), then $\sim(\gamma*\beta)*(\alpha*\gamma)=0$.

F) $\sim\beta*\alpha=0$ *implies* $\sim(\gamma*\beta)*(\alpha*\gamma)=0$.

Suppose that $\sim\alpha*\beta=0$, $\sim\gamma*\delta=0$, by F), we have

$$\sim(\delta*\alpha)*(\beta*\delta)=0,$$
$$\sim(\alpha*\gamma)*(\delta*\alpha)=0.$$

Then by C), $(\beta*\delta)*\sim(\alpha*\gamma)=0$. Hence

G) $\sim\alpha*\beta=0$, $\sim\gamma*\delta=0$ *imply* $(\beta*\delta)*\sim(\alpha*\gamma)=0$.

In F), if we put $\alpha=\sim\sim p$, $\beta=p$, $\gamma=r$, then by 5),

8) $\sim(r*p)*(\sim\sim p*r)=0$.

For any expression α, by 7), we have $\sim\sim\sim\sim\alpha*\sim\alpha=0$, hence

H) $\alpha=0$ implies $\sim\sim\alpha=0$.

The following propositions are fundamental for our discussion.

I) $\sim\beta*\alpha=0$, $\sim\gamma*\beta=0$, $\sim\delta*\gamma=0$ imply $\sim\delta*\alpha=0$.

Proof. By H) and $\sim\gamma*\beta=0$, we have $\sim\sim(\sim\gamma*\beta)=0$. On the other hand $\sim\delta*\gamma=0$ and 6) imply $\sim\sim\gamma*\sim\delta=0$. From this and

$\sim\beta*\alpha=0$, we have

$$(\sim\delta*\alpha)*\sim(\sim\gamma*\beta)=0$$

By G). By D), we have

$$\sim\sim(\sim\delta*\alpha)*\sim\sim\sim(\sim\gamma*\beta)=0.$$

Then, by $\sim\gamma*\beta=0$, and H), we have $\sim\sim(\sim\gamma*\beta)=0$, and hence from 4), $\sim\delta*\alpha=0$. Therefore we complete the proof.

9) $\sim p*p=0$.

Proof. The idea of the proof is due to B. J. Rosser ([2], p. 64). In 1), put $p=\sim\sim p$,

$$\sim(\sim\sim p*\sim\sim p)*\sim\sim p=0.$$

In 8), pur $r=\sim\sim p, r=p$, then we have

$$\sim(\sim\sim p*p)*(\sim\sim p*\sim\sim p)=0,$$

$$\sim(p*p)*(\sim\sim p*p)=0$$

respectively. Hence by I), we have

(1) $$\sim(p*p)*\sim\sim p=0$$

From 2), we have

(2) $$\sim p*(p*p)=0.$$

(1), (2), and C) imply

$$\sim\sim p*\sim p=0.$$

Hence by E), we have $\sim p*p=0$, which completes the proof.

$\sim p*p=0$ and F) imply

10) $\sim(r*p)*(p*r)=0$.

In 9), put $\delta=\gamma$, then

J) $\sim\beta*\alpha=0, \sim\gamma*\beta=0$ *imply* $\sim\gamma*\alpha=0$.

From 10) and 2), we have

(3) $$\sim(p*q)*(q*p)=0,$$

(4) $$\sim q*(p*q)=0$$

respectively. (3), (4), and J) imply

11) $\sim q*(q*p)=0$.

K) $\sim\beta*\alpha=0, \sim\delta*\gamma=0$ *imply* $\sim(\delta*\beta)*(\gamma*\alpha)=0$.

Proof. In 3), put $p=q$, then

(5) $$\sim\sim(p*r)*\sim(r*p)=0.$$

On the other hand, $\sim\beta*\alpha=0, \sim\delta*\gamma=0$, and G) imply

(6) $$(\gamma*\alpha)*\sim(\delta*\beta)=0.$$

Put $r=\gamma*\alpha, p=\sim(\delta*\beta)$ in (5), then

$$\sim\sim(\sim(\delta*\beta)*(\gamma*\alpha))*\sim((\gamma*\alpha)*\sim(\delta*\beta))=0.$$

By (6) and 4), we have

$$\sim(\delta*\beta)*(\gamma*\alpha)=0,$$

which completes the proof.

Next we shall prove a fundamental proposition.

12) $\sim(p^*\sim\sim q)*(p*q)=0$.

Proof. By 7), 9), we have $\sim\sim\sim q*q=0, \sim p*p=0$ respectively. Applying K), then

$$\sim(p*\sim\sim q)*(p*q)=0.$$

13) $\sim(\sim q*p)*(\sim(r*q)*(p*r))=0.$

Proof. From 3) and E), we have

$$\sim(\sim q*p)*(\sim\sim(p*r)*\sim(r*q))=0. \tag{13}$$

By 10), we have

$$\sim(\sim\sim(p*r)*\sim(r*q))*(\sim(r*q)*\sim\sim(p*r))=0. \tag{14}$$

Further, by 12), we have

$$\sim(\sim(r*q)*\sim\sim(p*r))*(\sim(r*q)*(p*r))=0. \tag{15}$$

Hence, by (13), (14), (15), and I), we have

$$\sim(\sim(q*p))*(\sim(r*q)*(p*r))=0,$$

which completes the proof.

Among the propositions proved above, the propositions 1), 6), 11), and 13) form an axiom system by B. Sobociński [4]. The proof of the converse is given in [4].

References

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