250. *On Certain Condition for the Principle of Limiting Amplitude*

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1. Introduction and results. We consider the nonstationary problems

$$\left[\frac{\partial^2}{\partial t^2}-\Delta+q(x)\right]u(x, t)=f(x)e^{-i\sqrt{\lambda}t} \qquad (\lambda>0), \tag{1}$$

$$u(x, 0)=0, \qquad \frac{\partial}{\partial t}u(x, 0)=0; \tag{2$'$}$$

$$\left[\frac{\partial^2}{\partial t^2}-\Delta+q(x)\right]u(x, t)=0, \tag{1$'$}$$

$$u(x, 0)=g_1(x), \qquad \frac{\partial}{\partial t}u(x, 0)=g_2(x); \tag{2}$$

in 3 Euclidean space R^3, where $q(x)$ is a real-valued function belonging to $C_0^2(R^3)$. Furthermore assume that the operator $L=-\Delta+q(x)$ has no eigenvalue. Here Δ denotes the Laplacian $\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\partial^2/\partial x_3^2$, and L is the unique self-adjoint extension in $L^2(R^3)$ of $-\Delta+q$ defined on $C_0^\infty(R^3)$. Then under the conditions imposed on q, L is strictly positive, and it is known that $D(L)=W_2^2(R^3)$, where $W_2^2(R^3)$ denotes the space of functions whose partial derivatives of order $\leqq 2$ in the sense of distribution belong to $L^2(R^3)$.

Then we have the following

Theorem 1. *Suppose that* $g_1(x)\in C_0^2(R^3)$, $g_2(x)\in C_0^1(R^3)$, *and* $f(x)\in C_0^1(R^3)$. *Then the following three conditions are equivalent*:

i) *The solution of the problem* (1), (2)′ *is such that at every point* $x\in R^3$ *we have*

$$\lim_{t\to\infty} u(x, t)e^{i\sqrt{\lambda}t}=u_+(x, \lambda) \qquad (\lambda>0),$$

where $u_+(x, \lambda)$ *denotes* $\lim_{\varepsilon\to+0} u_\varepsilon(x, \lambda)$ *and* $u_\varepsilon(x, \lambda)$ *is the solution of the equation*

$$Lu=(\lambda+i\varepsilon)u+f.$$

ii) *The solution of the problem* (1)′, (2) *is such that at every point* $x\in R^3$ *we have*

$$\lim_{t\to\infty} u(x, t)=0.$$

iii) *Every solution of the equation* $(-\Delta+q)u=0$, *satisfying the conditions* $u=O(|x|^{-1})$, $\dfrac{\partial u}{\partial x_k}=O(|x|^{-2})$ *at infinity is identically zero* (cf. [4]).

For the special case where $q(x)$ depends only on $|x|$ and satisfies the inequality

$$-q(x) \leqq \left(\frac{1}{4}+2\right)\frac{1}{|x|^2},$$

we give the relation of the principle of limit amplitude and the characteristics.

Theorem 2. *If there exists a solution $u(x)$ of the equation $(-\Delta+q)u=0$ which is not identically zero and satisfies the conditions $u=O(|x|^{-1})$, $\frac{\partial u}{\partial x_k}=O(|x|^{-2})$ at infinity, then there exists a solution $v(x, t)$ of the problem*

$$\frac{\partial^2}{\partial t^2}v+Lv=0,$$

$$v(x, 0)=g_1(x), \qquad \frac{\partial}{\partial t}v(x, 0)=g_2(x),$$

such that $v(x, t)=u(x)$ for $|x|\leqq t$, where $g_1(x)\in C_0^2(R^3)$, $g_2(x)\in C_0^1(R^3)$.

2. Proof of Theorem 1. From theorem 6 in [1] it follows that iii) implies i) and ii). To prove the converse assertion in Theorem 1 we use the following Lemmas together with the methords considered in [1] or [3].

Lemma 2 (Fredholm). *Let $q(x)=0$ for $|x|>r_0$ and $R(x, y, \lambda)$ be the resolvent kernel of the equation*

$$u(x)=\int -\frac{1}{4\pi}\frac{e^{-\lambda|x-y|}}{|x-y|}q(y)u(y)dy+\psi(x)$$

in Ω, where Ω is a compact set of R^3. Then we see that $R(x, y, \lambda)$ has the form

$$R(x, y, \lambda)=\frac{w(x)v_m(y)}{(\lambda-\lambda_0)^m}+\cdots+\frac{w(x)v_1(y)}{(\lambda-\lambda_0)}+K(x, y, \lambda)$$

in a neighbourhood of a pole λ_0 of $R(x, y, \lambda)$, where $v_j(y)$, $(j=1, 2, \cdots, m)$ are non-trivial solutions of the equation

$$v(y)=-\frac{1}{4\pi}q(y)\int\frac{1}{|y-s|}v(s)ds \quad \text{in } \Omega,$$

and $K(x, y, \lambda)$ is continuous in (x, y, λ) for $x\neq y$, analytic in λ for $x\neq y$, $K(x,y,\lambda)=0$ for $|y|>r_0$ and $K(x,y,\lambda)=O(|x-y|^{-1})$ as $|x-y|\to 0$.

Let E_λ be the resolution of the identity generated by the operator L. Since $E_{\lambda+0}=0$, we have

Lemma 2. *If there exists a solution of $(-\Delta+q)w=0$ which is not identically zero and $w=O(|x|^{-1})$, $\frac{\partial w}{\partial x_k}=O(|x|^{-2})$ at infinity, then in Lemma 1 we have that $\lambda_0=0$ and $m=1$.*

3. Proof of Theorem 2. It follows from [4] that $u(x)$ depends only on $|x|$. Set $w(r)=ru(x)$, where $r=|x|$. Then we have

$$\frac{d^2}{dt^2}w(r)-q(r)w(r)=0 \quad \text{for } r\geqq 0,$$

where $q(r)=q(x)$, $q(r)=0$ for $r>r_0$.

If we set $u(r,t)=w(r)$ in D_1, where $D_1\equiv\{(r,t);\, 0\leqq r\leqq t\}$, we see that $u(r,t)$ satisfies the equation

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}+q\right]u(r,t)=0 \tag{3}$$

in D_1 with the condition: $u(0,t)=0$.

Next in $D_2\equiv\{(r,t);\, 0\leqq t\leqq r\}$ we shall find the solution $u_1(r,t)$ of the equation (3) with initial deta: $u_1(r,0)=\varphi_1(r)$, $\frac{\partial u_1}{\partial t}(r,0)=\varphi_2(r)$, where $\varphi_1\in C^3([0,\infty))$, supp $\varphi_1\subset[0,2r_0]$, $\varphi_2\in C^2([0,\infty))$ and supp $\varphi_2\subset(2r_0,r_1)$, where $r_1>2r_0$. Furthermore we see that $u_1(r,t)$ satisfies the integral equation

$$u_1(r,t)=\frac{1}{2}\{\varphi_1(r+t)+\varphi_1(r-t)\}+\frac{1}{2}\int_{r-t}^{r+t}\varphi_2(s)ds$$
$$+\frac{1}{2}\int_0^t d\tau\int_{r-(t-\tau)}^{r+(t-\tau)}-q(s)u_1(s,\tau)ds \tag{4}$$

in D_2. Since supp $\varphi_2\subset(2r_0,r_1)$, we see that $\tilde{u}_1(r,t)\equiv\frac{1}{2}\int_{r-t}^{r+t}\varphi_2(s)ds$ satisfies (3) in D_2. Therefore for $r>\frac{r_1}{2}$ we can assume that $w(r)=u_1(r,r)=C$, where C is a constant and $C\neq 0$.

Now we shall find the solution $u_2(r,t)$ of (3) in D_2 satisfying the initial deta with compact supports such that $u_2(r,r)=w(r)-u_1(r,r)$ for $r\geqq 0$.

To this end we replace the coordinates r,t by new coordinates ξ_1,ξ_2 such that $\xi_1=\frac{1}{2}(r+t)$, $\xi_2=\frac{1}{2}(r-t)$. In $D_3\equiv\{(\xi_1,\xi_2);\ \xi_1\geqq 0$, and $\xi_2\geqq 0\}$, we consider

$$\tilde{u}(\xi_1,\xi_2)=\psi(\xi_1)-\int_0^{\xi_2}d\zeta_2\int_{\xi_1}^{r_0}q(\zeta_1+\zeta_2)\tilde{u}(\zeta_1,\zeta_2)d\zeta_1, \tag{5}$$

where $\psi(\xi_1)=w(r)-u_1(r,r)\in C^3([0,\infty))$ and $\psi(\xi_1)=0$ for $\xi_1>\frac{r_1}{2}$. By $\bar{C}(D_3)$ we denote the Banach space of all bounded continuous functions $\tilde{v}(\xi_1,\xi_2)$ defined on D_3, with the norm $\|\tilde{v}\|=\sup_{(\xi_1,\xi_2)\in D_3}|\tilde{v}(\xi_1,\xi_2)|$. Instead of (5) we consider the equation

$$\tilde{v}(\xi_1,\xi_2)=e^{\alpha(\xi_1-\xi_2)}\psi(\xi_1)-e^{\alpha(\xi_1-\xi_2)}\int_0^{\xi_2}d\zeta_2\int_{\xi_1}^{r_0}q(\zeta_1+\zeta_2)e^{-\alpha(\zeta_1-\zeta_2)}\tilde{v}(\zeta_1,\zeta_2)d\zeta_1, \tag{6}$$

where α is an arbitrary positive number. Set

$$(T_\alpha\tilde{v})(\xi_1,\xi_2)=-e^{\alpha(\xi_1-\xi_2)}\int_0^{\xi_2}d\zeta_2\int_{\xi_1}^{r_0}q(\zeta_1+\zeta_2)e^{-\alpha(\zeta_1-\zeta_2)}\tilde{v}(\zeta_1,\zeta_2)d\zeta_1.$$

Then T_α is a continuous linear operator on $\bar{C}(D_3)$ to $\bar{C}(D_3)$ and we have

$$\| T_\alpha \| \leqq \frac{1}{2\alpha}\int_0^{r_0} | q(r) | \, dr.$$

Therefore for sufficiently large α there exists a unique solution $\tilde{v}(\xi_1, \xi_2)$ of (6) belonging to $\bar{C}(D_3)$. Setting $\tilde{u}(\xi_1, \xi_2) = e^{-\alpha(\xi_1-\xi_2)}\tilde{v}(\xi_1, \xi_2)$, we see that $\tilde{u}(\xi_1,\xi_2)$ is continuous and satisfies (5) in D_3. Hence $\tilde{u}(\xi_1,\xi_2) \in C^3(D_3)$. Furthermore we have that $\tilde{u}(\xi_1, \xi_2) = 0$ for $\xi_1 > \frac{r_1}{2}$.

If we set

$$u_2(r, t) = \tilde{u}(\xi_1, \xi_2), \qquad g_1(r) = \varphi_1(r) + \tilde{u}\left(\frac{r}{2}, \frac{r}{2}\right),$$

$$g_2(r) = \varphi_2(r) + \frac{1}{2}\left\{\frac{\partial}{\partial \xi_1}\tilde{u}\left(\frac{r}{2}, \frac{r}{2}\right) - \frac{\partial}{\partial \xi_2}\tilde{u}\left(\frac{r}{2}, \frac{r}{2}\right)\right\},$$

from (5) we have

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + q\right] u_2(r, t) = 0 \quad \text{in } D_2,$$

$$u_2(r, r) = w(r) - u_1(r, r), \; u_2(r, 0) = g_1(r) - \varphi_1(r), \; \frac{\partial}{\partial t}u_2(r, 0) = g_2(r) - \varphi_2(r),$$

and furthermore

$$g_1(r) = g_2(r) = 0 \text{ for } r > r_1, \quad g_1(r) \in C^3([0, \infty)), \quad g_2(r) \in C^2([0, \infty)).$$

If we set

$$u_3(r, t) = u_1(r, t) + u_2(r, t) \quad \text{in } D_2,$$

then we have that $u_3(r, t)$ satisfies (3) in D_2 with the initial deta $u_3(r, 0) = g_1(r)$, $\frac{\partial}{\partial t}u_3(r, 0) = g_2(r)$ and on the characteristic line $\{(r, t); t = r\}$ we have $w(r) = u_3(r, r)$. Therefore if we set

$$u(r, t) = w(r) \quad \text{in } D_1,$$
$$= u_3(r, t) \quad \text{in } D_2,$$

then we have that in $\{(r, t); r \geqq 0 \text{ and } t \geqq 0\}$, $u(r, t)$ satisfies (3) in the sense of distribution and $u(0, t) = 0$, $u(r, 0) = g_1(r)$, $\frac{\partial}{\partial t}u(r, 0) = g_2(r)$, $u(r, t) = w(r)$ for $r \leqq t$.

Furthermore we have the following

Lemma 3. *We can take $g_1(r)$, $g_2(r)$ such that*

$$g_1(0) = g_2(0) = 0.$$

We shall postpone to prove Lemma 3. Setting $v(x, t) = r^{-1}u(r, t)$, $\bar{g}_1(x) = r^{-1}g_1(r)$, $\bar{g}_2(x) = r^{-1}g_2(r)$, by virtue of Lemma 3, we have $\bar{g}_1(x) \in C_0^2(R^3)$, $\bar{g}_2(x) \in C_0^1(R^3)$, and

$$\left[\frac{\partial^2}{\partial t^2} - \Delta + q\right] v(x, t) = 0,$$

$$v(x, 0) = \bar{g}_1(x), \qquad \frac{\partial}{\partial t}v(x, 0) = \bar{g}_2(x), \qquad v(x, t) = u(x) \quad \text{for } |x| \leqq t.$$

which is the assertion of Theorem 2.

Proof of Lemma 3. By the construction of $g_1(r)$, $g_2(r)$, we have

$$g_1(0)=w(0)=0,$$

$$g_2(0)=\frac{1}{2}-\varphi_2'(0)+\frac{1}{2}w'(0)+\frac{1}{2}\int_0^{r_0}q(r)w(r)dr-\frac{1}{2}\int_0^{r_0}q(r)u_1(r,r)dr.$$

Set

$$K=huw'(0)+hu\int_0^{r_0}q(r)w(r)dr.$$

Then we have $g_2(0)=0$ if and only if we have

$$\varphi_1'(0)+\int_0^{r_0}q(r)u_1(r,r)dr=K. \tag{7}$$

Let $\bar{u}_1(r,t)$ be the solution of (3) in D_2 with initial deta $\bar{u}_1(r,0)=\bar{\varphi}_1(r)$, $\frac{\partial}{\partial t}\bar{u}_1(r,0)=0$, where $\bar{\varphi}_1(r)\in C^3([0,\infty))$ and $\operatorname{supp}\bar{\varphi}_1\subset[0,2r_0]$. Then we can take $\bar{\varphi}_1(r)$ such that

$$\bar{\varphi}_1'(0)+\int_0^{r_0}q(r)\bar{u}_1(r,r)dr\neq 0.$$

In fact, we have

$$\bar{u}_1(r,r)=\frac{1}{2}\{\bar{\varphi}_1(2r)+\bar{\varphi}_1(0)\}+\frac{1}{2}\int_0^r d\tau\int_\tau^{2r-\tau}-q(s)\bar{u}_1(s,\tau)ds,$$

by virtue of (4). Since we have

$$\int_0^{r_0}d\tau\left(\int_0^{r_0}|\bar{u}_1(s,\tau)|^2ds\right)^{\frac{1}{2}}\leqq C\left(\int_0^{r_0}|\bar{\varphi}_1(s)|^2ds\right)^{\frac{1}{2}}$$

by virtue of the energy inequality, we have

$$\left|\int_0^{r_0}q(r)\bar{u}_1(r,r)dr\right|\leqq C'\sup|\bar{\varphi}_1(r)|,$$

where C' is a constant depending on q, r_0.
If we choose $\bar{\varphi}_1$ such that

$$|\bar{\varphi}_1'(0)|>C'\sup_{0\leq r\leq 2r_0}|\bar{\varphi}_1(r)|,$$

we have

$$\bar{\varphi}_1'(0)+\int_0^{r_0}q(r)\bar{u}_1(r,r)dr\neq 0.$$

If in (7) we replace $u_1(r,r)$ by $u_1(r,r)+k\bar{u}_1(r,r)$, then (7) becomes

$$\varphi_1'(0)+\int_0^{r_0}q(r)u_1(r,r)dr+k\left\{\bar{\varphi}_1'(0)+\int_0^{r_0}q(r)\bar{u}_1(r,r)dr\right\}=K, \tag{8}$$

where k is an arbitrary real number. Taking k such that (8) holds, we have $g_2(0)=0$ for the $g_2(r)$ which is obtained by replacing $u_1(r,t)$ by $u_1(r,t)+k\bar{u}_1(r,t)$.

Since $q(r)=0$ for $r>r_0$ and $w(r)$ is bounded, we see that $w(r)=\text{constant}$ for $r>r_0$, therefore set $w(r)=C$ for $r>r_0$. We also require that $k\bar{u}_1(r,r)+u_1(r,r)=C$ for $r>\frac{r_1}{2}$, that is,

$$\varphi_1(0)+k\bar{\varphi}_1(0)+\int_0^{r_1}\varphi_2(s)ds+\int_0^{r_0}d\tau\int_\tau^{r_0}-q(s)\{u_1(s,\tau)+k\bar{u}_1(s,\tau)\}ds$$
$$=2C \quad \text{for } r>\frac{r_1}{2}. \qquad (9)$$

But in the equality (9) the values of $u_1(s,\tau)+k\bar{u}_1(s,\tau)$ for $(s,\tau)\in\{(s,\tau); 0\leq\tau\leq s\leq r_0\}$ depend only on the values of $\varphi_1(r)$, $\varphi_2(r)$, $\bar{\varphi}_1(r)$ for $r\in[0, 2r_0]$, and are independent of the values of $\varphi_2(r)$ for $r\in(2r_0, r_1)$. Therefore it is obvious that we can take $\varphi_2(r)$ such that (9) holds for $r>\frac{r_1}{2}$. Q.E.D.

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