245. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XXV

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(Comm. by Kinjirô KUNUGI, M.J.A., Dec. 12, 1966)

In this paper, by applying the functional-representations of normal operators in Hilbert spaces to somewhat abstracted and generalized integral equations, we shall illustrate that the expansions of solutions of such integral equations can be discussed by using integral operators alone even if we do not give any analytic condition from which the expansions of their corresponding kernels can be deduced.

Definitions of notations. Let Δ be a Lebesgue σ-measurable set of finite or infinite measure in real m-dimensional Euclidean space R_m; let $L_2(\Delta, \sigma)$ be the Lebesgue functionspace; let $\{\varphi_\nu(x)\}_{\nu=1,2,3,\dots}$ and $\{\psi_\mu(x)\}_{\mu=1,2,3,\dots}$ be both incomplete orthonormal systems such that the union of them forms a complete orthonormal system in $L_2(\Delta, \sigma)$; let (β_{ij}) be an infinite bounded normal matrix with $\sum_{j=1}^{\infty}|\beta_{ij}|^2 \neq |\beta_{ii}|^2 > 0$ $(i=1, 2, 3, \cdots)$; let $(\beta_{ij}^{(p)})=(\beta_{ij})^p$ $(p=1, 2, 3, \cdots, n)$ where $\beta_{ij}^{(1)}=\beta_{ij}$ $(i, j=1, 2, 3, \cdots)$; let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be any infinite bounded sequence of complex scalars; and for any positive integer p with $1 \leqq p \leqq n$ and $h(x) \in L_2(\Delta, \sigma)$ let N_p be an integral operator defined by

$$N_p h(x)=\sum_{\nu=1}^{\infty} \lambda_\nu^p \int_\Delta h(y)\overline{\varphi_\nu(y)}d\sigma(y) \cdot \varphi_\nu(x) + c^p \sum_{\mu=1}^{\infty} \left\{ \int_\Delta h(y)\overline{\psi_\mu(y)}d\sigma(y) \cdot \sum_{j=1}^{\infty} \beta_{\mu j}^{(p)} \psi_j(x) \right\},$$

where c is an arbitrarily given complex constant.

Theorem 68. *Let $g(x)$ be an arbitrarily given function in the subspace $\mathfrak{M}$ determined by $\{\varphi_\nu(x)\}_{\nu=1,2,3,\dots}$, and let ζ_p $(p=1, 2, 3, \cdots, n)$ be the roots of the equation $\lambda^n+\sum_{p=1}^{n} a_p \lambda^{n-p}=0$ with complex coefficients a_p. Then the integral equation*

$$\text{(A)} \qquad \lambda^n f(x)+\sum_{p=1}^{n} a_p \lambda^{n-p} N_p f(x)=g(x) \qquad \left(\lambda \notin \bigcup_{p=1}^{n} \overline{\{\zeta_p \lambda_\nu\}_{\nu=1,2,3,\dots}}\right)$$

has a uniquely determined solution

$$f_\lambda(x)=\sum_{\nu=1}^{\infty} c_\nu \prod_{p=1}^{n} (\lambda-\zeta_p \lambda_\nu)^{-1} \varphi_\nu(x) \in \mathfrak{M} \qquad (\textit{for almost every } x \in \Delta),$$

where $c_\nu=\int_\Delta g(x)\overline{\varphi_\nu(x)}d\sigma(x)$ $(\nu=1, 2, 3, \cdots)$ and $\sum_{\nu=1}^{\infty}|c_\nu|^2<\infty$; and in addition, if the set $\{\lambda_\nu\}$ is everywhere dense on an open or a closed *rectifiable Jordan curve* Γ, $\mathfrak{D}_\xi$ *denotes any very small neighborhood of an arbitrarily given point* ξ *on one*, $\zeta_k\Gamma$, *of the curves* $\zeta_p\Gamma$ *as the closed sets* $\overline{\{\zeta_p\lambda_\nu\}_{\nu=1,2,3,\cdots}}$ $(p=1, 2, 3, \cdots, n)$, Δ_ξ *denotes* $\mathfrak{D}_\xi-\left[\bigcup_{p=1}^{n}\zeta_p\Gamma\right]$, *and* $h(x)$ *is an arbitrary element in* $\mathfrak{M}$ *such that it consists of all* $\varphi_\nu(x)\in\{\varphi_\nu(x)\}$, *then the function* $T(\lambda)=\int_\Delta f_\lambda(x)\overline{h(x)}d\sigma(x)\not\equiv 0$ *assumes in* Δ_ξ *every finite value, with the possible exception of at most two finite values, an infinite number of times.*

Proof. As is found immediately from the earlier discussions [1], N_1 is a bounded normal operator in the concrete Hilbert space $L_2(\Delta, \sigma)$ and N_p $(p=2, 3, \cdots, n)$ are identical with the iterated (bounded) normal operators N_1^p respectively. Hence the given integral equation (A) is rewritten in the form

$$\left[\lambda^n I+\sum_{p=1}^{n}a_p\lambda^{n-p}N_1^p\right]f(x)=g(x) \qquad \left(\lambda\notin\bigcup_{p=1}^{n}\overline{\{\zeta_p\lambda_\nu\}_{\nu=1,2,3,\cdots}}\right),$$

where I denotes the identity operator. On the other hand, it is seen from the hypothesis on ζ_p that

$$\lambda^n I+\sum_{p=1}^{n}a_p\lambda^{n-p}N_1^p=\prod_{p=1}^{n}(\lambda I-\zeta_p N_1);$$

and moreover it is clear that $\{\lambda_\nu\}$ is the point spectrum of N_1 and that $\varphi_\nu(x)$ is an eigenfunction of N_1 corresponding to λ_ν. If we now denote by $\{K(z)\}$ the complex spectral family of N_1 and by $f_\lambda(x)$ the solution of (A), then these results permit us to conclude that

$$\begin{aligned} f_\lambda(x)&=\left[\prod_{p=1}^{n}(\lambda I-\zeta_p N_1)^{-1}\right]g(x) \qquad \left(\lambda\notin\bigcup_{p=1}^{n}\overline{\{\zeta_p\lambda_\nu\}_{\nu=1,2,3,\cdots}}\right)\\ &=\int_{\{\lambda_\nu\}}\prod_{p=1}^{n}(\lambda-\zeta_p z)^{-1}dK(z)g(x)\\ &=\sum_{\nu=1}^{\infty}c_\nu\prod_{p=1}^{n}(\lambda-\zeta_p\lambda_\nu)^{-1}\varphi_\nu(x) \qquad \text{(for almost every } x\in\Delta), \end{aligned}$$

where $c_\nu=\int_\Delta g(x)\overline{\varphi_\nu(x)}d\sigma(x)$ $(\nu=1, 2, 3, \cdots)$ and $\sum_{\nu=1}^{\infty}|c_\nu|^2=\int_\Delta|g(x)|^2d\sigma(x)<\infty$. Here $f_\lambda(x)$ belongs to $\mathfrak{M}$ because of

$$\int_\Delta|f_\lambda(x)|^2d\sigma(x)\leqq M^2\int_\Delta|g(x)|^2d\sigma(x)<\infty$$

where $M=\sup_\nu\prod_{p=1}^{n}|\lambda-\zeta_p\lambda_\nu|^{-1}$.

Next we consider the case where $\{\lambda_\nu\}$ is everywhere dense on the bounded open curve Γ defined in the statement of the present theorem. We may and do assume without loss of generality that the point $\xi\in\zeta_k\Gamma$ defined before is not either of the two extremities A, B of the curve $\zeta_k\Gamma$. Let M_1 be the middle point of the segment $\widehat{A\xi}$ on $\zeta_k\Gamma$ and M_2 the middle point of the segment $\widehat{M_1\xi}$ on $\zeta_k\Gamma$. Repeating this procedure, we have an infinite sequence of points M_μ $(\mu=1, 2, 3, \cdots)\in\widehat{A\xi}$ tending to ξ; and similarly we can construct another infinite sequence of points M'_μ $(\mu=1, 2, 3, \cdots)\in\widehat{B\xi}$ tending to ξ. Now we denote by q_ω the least positive integer of ν in $\zeta_k\lambda_\nu$ belonging to $\{\zeta_k\lambda_\nu\}_{\nu\geqq j}\cap\widehat{M_{\omega-1}M_\omega}$ where j is an arbitrary positive integer. Setting $\omega=1,2,3,\cdots$ and $M_0=A$, we have an infinite sequence of $\zeta_k\lambda_{q_\omega}$ $(\omega=1,2,3,\cdots)\in\widehat{A\xi}$ tending to ξ. In a similar way, we can construct another infinite sequence of points $\zeta_k\lambda_{q'_\omega}$ $(\omega=1,2,3,\cdots)\in\widehat{B\xi}$ tending to ξ. If we here put

$$f_\lambda^{[j]}(x)=\sum_{\nu=1}^{j-1}c_\nu\prod_{p=1}^{n}(\lambda-\zeta_p\lambda_\nu)^{-1}\varphi_\nu(x)+\sum_{\omega=1}^{\infty}c_{q_\omega}\prod_{p=1}^{n}(\lambda-\zeta_p\lambda_{q_\omega})^{-1}\varphi_{q_\omega}(x)$$
$$+\sum_{\omega=1}^{\infty}c_{q'_\omega}\prod_{p=1}^{n}(\lambda-\zeta_p\lambda_{q'_\omega})^{-1}\varphi_{q'_\omega}(x)$$

and then consider the function

$$T_j(\lambda)=\int_\Delta f_\lambda^{[j]}(x)\overline{h(x)}d\sigma(x)\not\equiv 0$$
$$\left(h(x)=\sum_{\nu=1}^{\infty}b_\nu\varphi_\nu(x)\in\mathfrak{M},\qquad b_\nu\neq 0\ (\nu=1,2,3,\cdots)\right),$$

every point belonging to the set

$$\bigcup_{p=1}^{n}\{\zeta_p\lambda_\nu\}_{\nu\leqq j-1}\cup\left[\bigcup_{p=1}^{n}\{\zeta_p\lambda_{q_\omega}\}_{\omega\geqq 1}\right]\cup\left[\bigcup_{p=1}^{n}\{\zeta_p\lambda_{q'_\omega}\}_{\omega\geqq 1}\right]$$

is a pole in the sense of the classical function theory; and for $T(\lambda)$ defined in the statement of the theorem we have

$$|\,T(\lambda)-T_j(\lambda)\,|=|\,(f_\lambda-f_\lambda^{[j]},h)\,|\quad\left(\lambda\notin\bigcup_{p=1}^{n}\{\zeta_p\lambda_\nu\}_{\nu=1,2,3,\cdots}\right)$$
$$=\left|\left(\sum\nolimits_{j\leqq\nu\neq q_\omega,q'_\omega(\omega=1,2,3,\cdots)}c_\nu\prod_{p=1}^{n}(\lambda-\zeta_p\lambda_\nu)^{-1}\varphi_\nu,\,h\right)\right|$$
$$\leqq\sum\nolimits_{j\leqq\nu\neq q_\omega,q'_\omega(\omega=1,2,3,\cdots)}|\,c_\nu\bar{b}_\nu\,|\,\left|\prod_{p=1}^{n}(\lambda-\zeta_p\lambda_\nu)\right|^{-1}$$
$$\leqq M\sum\nolimits_{j\leqq\nu\neq q_\omega,q'_\omega(\omega=1,2,3,\cdots)}|\,c_\nu\bar{b}_\nu\,|,$$

where M is the same notation as before and the right-hand series converges to zero as j becomes infinite, because of the fact that

$$\sum_{\nu=1}^{\infty}|c_\nu\bar{b}_\nu|\leqq\left\{\int_\Delta|\,g(x)\,|^2d\sigma(x)\right\}^{\frac{1}{2}}\left\{\int_\Delta|\,h(x)\,|^2d\sigma(x)\right\}^{\frac{1}{2}}<\infty.$$

This result shows that in the entire complex λ-plane $T(\lambda)$ is the limit function of $T_j(\lambda)$; and any $T_j(\lambda)$ has ξ as an accumulation point of polesin the sense of the classical function theory. In consequence, by reasoning exactly like that used to prove Theorem 41 [2] we can establish the latter half of the present theorem.

Similarly we can attain to the same conclusion for the case where Γ is closed.

Remark 1. If $\lambda_1=\lambda_2=\cdots=\lambda_m\neq\lambda_\nu$ $(\nu\geqq m+1)$ and $\lambda_1\neq 0$, and if $\zeta_k\lambda_1$ is not an accumulation point of $\bigcup_{p=1}^{n}\{\zeta_p\lambda_\nu\}_{\nu=1,2,3,\dots}$, then a necessary and sufficient condition that

$$\text{(B)}\qquad(\zeta_k\lambda_1)^n f(x)+\sum_{p=1}^{n}a_p(\zeta_k\lambda_1)^{n-p}N_1^p f(x)=g(x)\qquad(g(x)\in\mathfrak{M})$$

be solvable is that the equalities $\int_{\Delta} g(x)\overline{\varphi_j(x)}d\sigma(x)=0$ $(j=1, 2, 3, \cdots, m)$ hold. Let now this condition be fulfilled. Then the general solution of (B) is given (in the sense of convergence in mean) by

$$\sum_{\nu=m+1}^{\infty} c_\nu \prod_{p=1}^{n} (\zeta_k\lambda_1-\zeta_p\lambda_\nu)^{-1}\varphi_\nu(x)+\sum_{j=1}^{m} d_j\varphi_j(x)$$

where the d's are arbitrary finite complex scalars, as will be verified from the facts that $N_1\varphi_j(x)=\lambda_1\varphi_j(x)$ $(j=1,2,3,\cdots,m)$ and $\zeta_k^n+\sum_{p=1}^{n} a_p\zeta_k^{n-p}=0$.

Theorem 69. *Let* N_p $(p=1, 2, 3, \cdots, n)$, $\{\varphi_\nu(x)\}$, $\{\lambda_\nu\}$, $\mathfrak{M}$, *and* $g(x)$ *be the same notations as before; let*

$$\text{(C)}\qquad \begin{cases} \lambda_1=\lambda_2=\cdots=\lambda_{m_1}\neq\lambda_\nu \ (\nu\geqq m_1+1) \\ \lambda_{m_1+1}=\lambda_{m_1+2}=\cdots=\lambda_{m_2}\neq\lambda_\nu \ (\nu\geqq m_2+1) \\ \vdots \\ \lambda_{m_\alpha+1}=\lambda_{m_\alpha+2}=\cdots=\lambda_{m_n}\neq\lambda_\nu \ (\nu\geqq m_n+1); \end{cases}$$

and let $\lambda_{m_1}, \lambda_{m_2}, \cdots, \lambda_{m_n}$ *be not accumulation points of* $\{\lambda_\nu\}_{\nu>m_n}$ *and be the roots of the characteristic equation* $\rho^n+a_1\rho^{n-1}+a_2\rho^{n-2}+\cdots+a_n=0$ *for the given integral equation*

$$\text{(D)}\qquad (N_n+a_1N_{n-1}+a_2N_{n-2}+\cdots+a_nI)f(x)=g(x)\in\mathfrak{M}.$$

Then the validity of the chain of equalities $\int_{\Delta} g(x)\overline{\varphi_j(x)}d\sigma(x)=0$ $(j=1, 2, 3, \cdots, m_n)$ *is a necessary and sufficient condition for the existence of solutions of* (D); *and furthermore, if this condition is fulfilled, the general solution of* (D) *is given (in the sense of convergence in mean) by*

$$\text{(E)}\quad f(x)=\sum_{\nu=m_n+1}^{\infty} c_\nu \prod_{p=1}^{n} (\lambda_\nu-\lambda_{m_p})^{-1}\varphi_\nu(x)+\sum_{j=1}^{m_n} d_j\varphi_j(x)\ \left(c_\nu=\int_{\Delta} g(x)\overline{\varphi_\nu(x)}d\sigma(x)\right),$$

where the d's *are arbitrary complex scalars.*

Proof. Since, by hypotheses, (D) is rewritten

$$\prod_{p=1}^{n} (N_1-\lambda_{m_p}I)f(x)=g(x)$$

and $N_1-\lambda_{m_p}I$ $(p=1, 2, 3, \cdots, n)$ have no inverses respectively, it is obvious that the validity of the chain of equalities $\int_{\Delta} g(x)\overline{\varphi_j(x)}d\sigma(x)=0$ $(j=1, 2, 3, \cdots, m_n)$ is necessary for the existence of a solution of (D). We now suppose that this condition is fulfilled. Then $g(x)\in\mathfrak{M}$ is expressed in the form $g(x)=\sum_{\nu=m_n+1}^{\infty} c_\nu\varphi_\nu(x)$ where $c_\nu=\int_{\Delta} g(x)\overline{\varphi_\nu(x)}d\sigma(x)$, and hence a solution $f(x)$ of (D) is given by

$$\begin{aligned} f(x)&=\int_{\{\lambda_\nu\}_{\nu\geqq m_n+1}} \prod_{p=1}^{n} (z-\lambda_{m_p})^{-1}dK(z)g(x) \\ &=\sum_{\nu=m_n+1}^{\infty} c_\nu \prod_{p=1}^{n} (\lambda_\nu-\lambda_{m_p})^{-1}\varphi_\nu(x). \end{aligned}$$

In addition, there is no difficulty in showing that the equality

$$\prod_{p=1}^{n}(N_1-\lambda_{m_p}I)\sum_{j=1}^{m_n}d_j\varphi_j(x)=0$$

holds almost everywhere on Δ for any finite complex constants d_j ($j=1, 2, 3, \cdots, m_n$) and that, for any function $k(x)$ belonging to the subspace $\mathfrak{N}$ determined by the orthonormal set $\{\psi_\mu(x)\}$ defined at the beginning of this paper, $(N_n+a_1N_{n-1}+a_2N_{n-2}+\cdots a_nI)k(x)$ also belongs to $\mathfrak{N}$. Consequently the general solution of (D) is given by (E) for almost every $x\in\Delta$.

With these results, the proof of the theorem is complete.

By reasoning like that used above, we can easily establish the two following theorems:

Theorem 70. *Suppose that, as before, the isolated points* $\lambda_{m_p}\in\{\lambda_\nu\}$ ($p=1, 2, 3, \cdots, s; s<n$) *satisfying* (C) *are roots of the characteristic equation for* (D) *and that* $\zeta_p\notin\overline{\{\lambda_\nu\}}$ ($p=s+1, s+2, \cdots, n$) *are the other roots of the same equation. Then a necessary and sufficient condition that* (D) *be solvable is that the equalities* $\int_\Delta g(x)\overline{\varphi_j(x)}d\sigma(x)=0$ ($j=1, 2, 3, \cdots, m_s$) *hold; and if this condition is fulfilled, the general solution of* (D) *is given (in the sense of convergence in mean) by*

$$f(x)=\sum_{\nu=m_s+1}^{\infty}c_\nu\prod_{p=1}^{s}(\lambda_\nu-\lambda_{m_p})^{-1}\prod_{p=s+1}^{n}(\lambda_\nu-\zeta_p)^{-1}\varphi_\nu(x)+\sum_{j=1}^{m_s}d_j\varphi_j(x)\in\mathfrak{M}$$

$$\left(c_\nu=\int_\Delta g(x)\overline{\varphi_\nu(x)}d\sigma(x)\right),$$

where the d's are arbitrary complex scalars.

Theorem 71. *Let any root* ζ_p *of the characteristic equation for* (D) *be not a point of* $\overline{\{\lambda_\nu\}}$. *Then* (D) *has a uniquely determined solution*

$$f(x)=\sum_{\nu=1}^{\infty}c_\nu\prod_{p=1}^{n}(\lambda_\nu-\zeta_p)^{-1}\varphi_\nu(x)\in\mathfrak{M}\qquad\left(c_\nu=\int_\Delta g(x)\overline{\varphi_\nu(x)}d\sigma(x)\right)$$

in the sense of convergence in mean.

Remark 2. If the characteristic equation of (D) has at least one root such that it is an accumulation point of $\{\lambda_\nu\}$, clearly (D) has no solution as far as $g(x)\in\mathfrak{M}$ contains all $\varphi_\nu(x)$ except a finite number of elements belonging to $\{\varphi_\nu(x)\}$.

Theorem 72. *Let one,* ζ_1, *of the roots* ζ_p ($p=1, 2, 3, \cdots, n$) *of the characteristic equation for* (D) *be an accumulation point of* $\{\lambda_\nu\}$; *let the others* ζ_p ($p=2, 3, \cdots, n$) *be not on the closure* $\overline{\{\lambda_\nu\}}$; *let* $\{\lambda_{k_\nu}\}_{\nu=1,2,3,\ldots}$ *be all those elements of* $\{\lambda_\nu\}$ *which lie on the disc* $\{\lambda: |\lambda-\zeta_1|\leqq\varepsilon\}$ *for an arbitrarily given small positive* ε; *and let* $g(x)$ *in* (D) *be given by*

$$g(x)=\sum_{\nu=1}^{\infty}c_\nu\varphi_\nu(x)-\sum_{\nu=1}^{\infty}c_{k_\nu}\varphi_{k_\nu}(x)\equiv\sum_{\nu=1}^{\infty}c_{n_\nu}\varphi_{n_\nu}(x)\qquad\left(c_\nu=\int_\Delta g(x)\overline{\varphi_\nu(x)}d\sigma(x)\right).$$

Then if ζ_1 *does not belong to* $\{\lambda_\nu\}$ *itself,* (D) *has a uniquely determined* *solution*

(F) $\quad f(x)=\sum_{\nu=1}^{\infty} c_{n_\nu} \prod_{p=1}^{n} (\lambda_{n_\nu}-\zeta_p)^{-1}\varphi_{n_\nu}(x) \in \mathfrak{M}$ *(for almost every* $x \in \Delta$*);*

but if, contrary to it, ζ_1 *is* λ_α *belonging to* $\{\lambda_\nu\}$ *itself,* (D) *has (in the sense of convergence in mean) the general solution*

(G) $\quad f(x)=\sum_{\nu=1}^{\infty} c_{n_\nu} \prod_{p=1}^{n} (\lambda_{n_\nu}-\zeta_p)^{-1}\varphi_{n_\nu}(x)+\sum_{j=\alpha}^{\alpha+s} d_j\varphi_j(x) \in \mathfrak{M} \quad (\alpha+s<\infty),$

where $\varphi_\alpha(x), \varphi_{\alpha+1}(x), \varphi_{\alpha+2}(x), \cdots, \varphi_{\alpha+s}(x)$ *denote all the normalized eigenfunctions of* N_1 *corresponding to the eigenvalue* λ_α *and the d's are arbitrary complex scalars.*

Proof. In fact, it is readily found that, for $f(x)$ defined by (F),

$$\prod_{p=1}^{n}(N_p-\zeta_p I)f(x)=\int_{\{\lambda_\nu\}_{\nu\geqq 1}} \prod_{p=1}^{n}(z-\zeta_p)dK(z)f(x)$$
$$=\int_{\{\lambda_{n_\nu}\}_{\nu\geqq 1}} \prod_{p=1}^{n}(z-\zeta_p)dK(z)f(x)$$
$$=\sum_{\nu=1}^{\infty} c_{n_\nu}\varphi_{n_\nu}(x) \quad \text{(for almost every } x \in \Delta).$$

Moreover, if ζ_1 is not an eigenvalue of N_1, it is obvious by hypotheses that $\prod_{p=1}^{n}(N_1-\zeta_p I)\varphi_\nu(x)$ is not zero almost everywhere on Δ for any ν. Since, on the other hand, for any element $k(x)$ of the orthogonal complement $\mathfrak{N}$ of $\mathfrak{M}$ in $L_2(\Delta, \sigma)$ $\prod_{p=1}^{n}(N_1-\zeta_p I)k(x)$ also belongs to $\mathfrak{N}$ itself, the results just established imply the validity of the former half of the present theorem.

Next we suppose that ζ_1 is λ_α in $\{\lambda_\nu\}$ and that $\lambda_\alpha=\lambda_{\alpha+1}=\lambda_{\alpha+2}=\cdots=\lambda_{\alpha+s}\neq\lambda_\nu$ $(\nu=1, 2, \cdots, \alpha-1, \alpha+s+1, \alpha+s+2, \cdots)$. Then the equality

$$\prod_{p=1}^{n}(N_1-\zeta_p I)\sum_{j=\alpha}^{\alpha+s} d_j\varphi_j(x)=0$$

always holds almost everywhere on Δ for any system of finite complex constants d_j $(j=\alpha$ to $\alpha+s)$, while $\prod_{p=1}^{n}(N_1-\zeta_p I)\varphi_\nu(x)$ never vanishes almost everywhere on Δ for any $\varphi_\nu(x)$ different from $\varphi_j(x)$ $(j=\alpha$ to $\alpha+s)$. Moreover, even in this case $\prod_{p=1}^{n}(N_1-\zeta_p I)k(x)$ has the same property as that stated above. Consequently the general solution of (D) is given by $f(x)$ in (G). The latter half of the theorem has thus been proved.

References

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