242. Subdirectly Irreducible Infinite Bands: An Example

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A family $(\varphi_i)_{i\in I}$ of homomorphisms of a semigroup S into semigroups belonging to a class K is called an approximation of S in K if for every $s_1, s_2 \in S$, $s_1 \neq s_2$ there exists $i \in I$ such that $\varphi_i(s_1) \neq \varphi_i(s_2)$. If $\varphi_i(S)$ are finite for all $i \in I$, then the approximation is called finite.

Clearly, approximations $(\varphi_i)_{i\in I}$ of S in K are in a natural 1-1 correspondence with isomorphisms φ of S into direct products of semigroups belonging to K (in fact, $\varphi = \Delta(\varphi_i)_{i\in I}$ where Δ denotes the semi-direct product of the second kind of mappings. This operation was introduced by V. V. Wagner [1]).

Approximations are also tightly connected with subdirect decompositions of semigroups, because a subdirect decomposition is exactly an approximation $(\varphi_i)_{i\in I}$ such that all φ_i are onto-homomorphisms. Evidently, if a semigroup S is subdirectly irreducible, then every approximation of S must contain an isomorphism. Hence an infinite subdirectly irreducible semigroup cannot possess a finite approximation.

Approximations of semigroups have been recently studied by M. M. Lesohin (see, for example, [2]) who raised the problem if every band (i.e., idempotent semigroup) has a finite approximation. Here we give the negative answer to this problem constructing an infinite subdirectly irreducible band.

Let N denote the set of all positive integers, a_i denote the constant mapping of N into itself: $a_i(n) = i$ for every $n \in N$. Let B be the set of all mappings b of N into itself such that: $b(1) = 1$, $b(2) = 2$, $b(n)$ is equal either to 1 or to 2 for every $n \in N$. Let $C = A \cup B$ where $A = (a_i)_{i \in N}$. If $x \in C$, then $a_i \circ x = a_i$, $x \circ a_i = a_{x(i)}$. It is easy to verify that if $x, y \in B$ then $y \circ x \in B$. Therefore C is a semigroup of transformations of N under natural multiplication $\circ$ of transformations. Evidently, each element of C is idempotent, so C is a band.

Define an equivalence relation ε_0 on C: $x \equiv y(\varepsilon_0)$ iff $x = y$ or $x, y \in \{a_1, a_2\}$. A straightforward verification proves ε_0 to be a congruence.

Let ε be a congruence on C, $x \equiv y(\varepsilon)$, $x \neq y$. Then there exists $n \in N$ such that $x(n) \neq y(n)$. Hence $a_{x(n)} = x \circ a_n \equiv y \circ a_n = a_{y(n)}$. If

$\{x(n), y(n)\}=\{1, 2\}$ then $\varepsilon_0 \subset \varepsilon$. Otherwise there exists $b \in B$ such that $\{b(x(n)), b(y(n))\}=\{1, 2\}$. But $a_{b(x(n))}=b \circ x \circ a_n \equiv b \circ y \circ a_n = a_{b(y(n))}$, therefore $\varepsilon_0 \subset \varepsilon$. Hence ε_0 is the least non-identical congruence on C.

Therefore C is an infinite subdirectly irreducible band. A is the core [3] of C. Our band C satisfies the left semi-normality property [4], that is, satisfies the following identity: $xyzx=xzyzx$. We do not know if there exists an infinite subdirectly irreducible band satisfying the identity $xyx=yx$.

References

[1] В. В. Вагнер: Полупрямое произведение бинарных отношений и бинарно разложимые полугруппы. Изв. вузов. Матем., **1**(32), 21-32 (1963).

[2] М. М. Лесохин: О гомоморфных представлениях полугрупп. Изв. вузов, Матем., **5**(48), 80-85 (1965).

[3] B. M. Schein [Saĭn]: Homomorphisms and subdirect decompositions of semigroups. Pacific J. Math., **17**, 529-547 (1966).

[4] N. Kimura: Note on idempotent semigroups. IV. Proc. Japan Acad., **34**, 121-123 (1958).