# 5. *Some Generalizations of V. Trnkova's Theorem on Unions of Strongly Paracompact Spaces*

By Yoshikazu YASUI

Department of Mathematics, Osaka Gakugei University

(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1967)

V. Trnkova [5] has recently investigated the unions of strongly paracompact spaces and he has proved the following interesting theorem:

*If space $X=X_1\cup X_2$, $X_1$, $X_2$ are closed and strongly paracompact subspaces, and the space $X_1\cap X_2$ has the locally Lindelöf property, then $X$ is itself strongly paracompact.* In this note, we shall obtain some generalizations of V. Trnkova's Theorem.

Let us quickly recall the definitions of terms which are used in this note. Let $X$ be a topological space, and $\mathfrak{N}$ be a collection of subsets of $X$. The collection $\mathfrak{N}$ is said to be *locally finite* if every point of $X$ has a neighborhood which intersects only finitely many elements of $\mathfrak{N}$. The collection $\mathfrak{N}$ is said to be *star finite* (resp. *star countable*) if each element of $\mathfrak{N}$ intersects only finitely (resp. only countably) many elements of $\mathfrak{N}$. Finally, $X$ is said to be *paracompact* (resp. *strongly paracompact*) if $X$ is Hausdorff and every open covering of $X$ has a locally finite open covering (resp. star finite open covering) of $X$ as a refinement.

**§1. Generalizations.** In this section, we shall get some generalizations of V. Trnkova's Theorem. At first, we shall show some lemmas.

**Lemma 1.** *Let $\mathfrak{B}=\{B_\alpha\mid\alpha\in A\}$ be a locally finite closed covering of a regular space $X$. If each $B_\alpha$ has the locally Lindelöf property as a subspace, then $X$ has the locally Lindelöf property.*

**Proof.** Let $x_0$ be an arbitrary point of $X$. Then, there exists a closed neighborhood $V_0(x_0)$ of $x_0$ in $X$ such that $V_0(x_0)$ intersects only all the members $B_{\alpha_1},\cdots,B_{\alpha_n}$ containing $x_0$. For each $i=1,2,\cdots,n$, by the locally Lindelöf property of $B_{\alpha_i}$, we have the closed neighborhood $V_i(x_0)$ of $x_0$ in $X$ such that $V_i(x_0)\cap B_{\alpha_i}$ has the Lindelöf property. Let $V=\bigcap_{i=0}^{n}V_i(x_0)$, then $V$ is a neighborhood of $x_0$ and $V=V\cap(\bigcup_{i=1}^{n}B_{\alpha_i})=\bigcup_{i=1}^{n}(V\cap B_{\alpha_i})$. This relation implies the Lindelöf property of $V$. Thus we get Lemma 1.

**Lemma 2.** *Let $\{F'_\alpha\mid\alpha\in A\}$ be a locally finite closed covering of a regular space $X$ where the index set $A$ is a well ordered set. If we define as follows: $F_1=F'_1$, $F_\alpha=\overline{F'_\alpha-\bigcup_{\beta<\alpha}F'_\beta}$ for each $\alpha>1$, then*

*$\{F_\alpha \mid \alpha\in A\}$ is a locally finite closed covering of $X$ such that $Q=\bigcup_{\alpha\neq\beta}(F_\alpha\cap F_\beta)\subset\bigcup_{\alpha\in A}\mathfrak{B}(F'_\alpha)$ where $\mathfrak{B}(F'_\alpha)$ denotes the boundary of $F'_\alpha$.*

**Proof.** It is obvious that $\{F_\alpha \mid \alpha\in A\}$ is a locally finite closed covering of $X$. Suppose that $x_0$ be an arbitrary element of $Q$. Then, $x_0\in F_\alpha\cap F_\beta$ for some $\alpha<\beta$, and hence $x_0\in F'_\alpha$. If $x_0\notin\mathfrak{B}(F'_\alpha)$, then there exists a neighborhood $V(x_0)$ contained in $F'_\alpha$ and hence $V(x_0)\subset\bigcup_{\gamma<\beta}F'_\gamma$. Then we get $x_0\notin F_\beta$, which is a contradiction.

By use of the above lemmas, we shall prove the following theorem which is a generalization of V. Trnkova's theorem.

**Theorem 1.** *Let $\mathfrak{F}'=\{F'_i \mid i=1,2,\cdots\}$ be a locally finite closed covering of a regular $T_1$-space $X$ such that each member $F'_i$ of $\mathfrak{F}'$ is a strongly paracompact subspace. If $\mathfrak{B}(F'_i)$ has the locally Lindelöf property for each $i=1,2,\cdots$, then $X$ is strongly paracompact.*

**Proof.** It is obvious that $X$ is paracompact. Now, let $F_1=F'_1$, $F_i=\overline{F'_i-\bigcup_{j<i}F'_j}$ for $i>1$ and $Q=\bigcup_{i\neq j}(F_i\cap F_j)$, then $\mathfrak{F}=\{F_i \mid i=1,2,\cdots\}$ is a locally finite closed covering of $X$ such that $Q\subset\bigcup_{i=1}^{\infty}\mathfrak{B}(F'_i)$ by Lemma 2, and $\bigcup_{i=1}^{\infty}\mathfrak{B}(F'_i)$ has the locally Lindelöf property by Lemma 1. On the other hand, it is easily seen that $Q$ is a closed subspace of $X$ and hence $Q$ is a paracompact subspace with the locally Lindelöf property. Therefore we can get the discrete covering $\mathfrak{G}=\{G_\lambda \mid \lambda\in\Lambda\}$ of $Q$ such that $G_\lambda$ has the Lindelöf property for each $\lambda\in\Lambda$ by V. Šediva [2]. In order to show the strong paracompactness of $X$, let $W$ be an arbitrary open covering of $X$, then it is sufficient to show that $\mathfrak{W}$ has a star countable open covering of $X$ as a refinement.

At first, we shall find the open covering $\mathfrak{U}$ of $X$ such that $\mathfrak{U}$ is a star refinement of $\mathfrak{W}$ and each member of $\mathfrak{U}$ intersects at most one element of $\mathfrak{G}$. For this purpose, let $\mathfrak{W}'=\{W_{\alpha\lambda} \mid \alpha\in A;\lambda\in\Lambda\}$, where $W_{\alpha\lambda}=W_\alpha\cap(G_\lambda\cup(X-Q))$, then $\mathfrak{W}'$ is an open covering of $X$ and the refinement of $\mathfrak{W}$.

Now, since $X$ is a regular $T_1$-space, $X$ is fully normal by A. H. Stone [4] and so there exists an open covering $\mathfrak{U}$ of $X$ such that $\mathfrak{U}$ is a star refinement of $\mathfrak{W}'$. Let $U$ be an arbitrary member of $\mathfrak{U}$ and so $U$ is contained in some member of $\mathfrak{W}'$, that is: $U\subset W_{\alpha_0\lambda_0}=W_{\alpha_0}\cap(G_{\lambda_0}\cup(X-Q))$ for some $\alpha_0\in A$, $\lambda_0\in\Lambda$, and therefore $U\cap Q\subset G_{\lambda_0}$. This implies that $U$ intersects at most one element of $G_{\lambda_0}$ of $\mathfrak{G}$ from the mutual disjointedness of $\{G_\lambda \mid \lambda\in\Lambda\}$. Thus we can get the open covering $\mathfrak{U}$ of $X$ such that $\mathfrak{U}$ is a star refinement of $\mathfrak{W}$ and each member of $\mathfrak{U}$ intersects at most one element of $\mathfrak{G}$.

Next, let $\mathfrak{U}_i=\mathfrak{U}\cap F_i$[1)] for each $i=1,2,\cdots$, then, there exists a

1) $\mathfrak{U}\cap F$ will denote the collection $\{U\cap F \mid U\in\mathfrak{U}\}$.

star countable covering $\mathfrak{B}_i$ of $F_i$ such that $\mathfrak{B}_i$ is a open collection in $F_i$ and a refinement of $\mathfrak{U}_i$ by the assumption. For each $i=1, 2, \cdots$, and each $\lambda \in \Lambda$, we can get a countable subcollection $\mathfrak{B}_{\lambda i}$ of $\mathfrak{B}_i$ such that $\mathfrak{B}_{\lambda i}$ is a covering of $G_\lambda \cap F_i$ from the Lindelöf property of $G_\lambda \cap F_i$, where we may assume that for each $V_i^\lambda$ of $\mathfrak{B}_{\lambda i}$, $V_i^\lambda \cap G_\lambda \cap F_i \neq \emptyset$ and hence $V_i^\lambda \cap Q \subset G_\lambda$. Still more, for each $\lambda \in \Lambda$, let $\mathfrak{B}_\lambda = \left\{ \operatorname{Int}\left(\bigcup_{i=1}^{n} V_{j(k_i)}^\lambda\right) \middle| V_{j(k_i)}^\lambda \in \mathfrak{B}_{\lambda k_i} \text{ for } i=1, 2, \cdots, n; \bigcap_{i=1}^{n} V_{j(k_i)}^\lambda \neq \emptyset; j(k_i)=1, 2, \cdots \text{ for } i=1, 2, \cdots, n; n=1, 2, \cdots \right\}$. Then $\mathfrak{B}_\lambda$ is evidently a countable open collection in $X$ and furthermore we shall show that this collection $\mathfrak{B}_\lambda$ is a covering of $G_\lambda$.

For this purpose, let $x_0$ be an arbitrary point of $G_\lambda$, then there exists a neighborhood $V(x_0)$ of $x_0$ in $X$ such that "$V(x_0) \cap F_j \neq \emptyset$" is equivalent to "$x_0 \in F_j$". Let $F_{i_1}, \cdots, F_{i_n}$ be all the members of $\mathfrak{F}$ containing $x_0$. For each $j=1, 2, \cdots, n$, $x_0 \in G_\lambda \cap F_{i_j}$ and hence there exists an open neighborhood $V_{i_j}'$ of $x_0$ in $X$ such that $x_0 \in V_{i_j}' \cap F_{i_j} \subset V_{i_j}$ for some $V_{i_j}$ of $\mathfrak{B}_{\lambda i_j}$. Let $G = V(x_0) \cap (\bigcap_{j=1}^{n} V_{i_j}')$, then $G$ is a neighborhood of $x_0$ in $X$ and $G \subset \bigcup_{j=1}^{n} V_{i_j}$ where $x_0 \in V_{i_j} \in \mathfrak{B}_{\lambda i_j}$. This means $x_0 \in \operatorname{Int}\left(\bigcup_{j=1}^{n} V_{i_j}\right)$ and $\operatorname{Int}\left(\bigcup_{j=1}^{n} V_{i_j}\right)$ is a member of $\mathfrak{B}_\lambda$. Lastly let $\mathfrak{H}_i = \{V - Q \mid V \in \mathfrak{B}_i - \bigcup_\lambda \mathfrak{B}_{\lambda i}\}$ for each $i=1, 2, \cdots$ and $\mathfrak{H} = (\bigcup_1^\infty \mathfrak{H}_i) \cup (\bigcup_\lambda \mathfrak{B}_\lambda)$. Then we shall show that this collection $\mathfrak{H}$ is a star countable open covering of $X$ and a refinement of $\mathfrak{W}$.

(1) *$\mathfrak{H}$ is an open family of $X$.* For this purpose, it suffices to show that $\mathfrak{H}_i$ is an open collection of $X$ for each $i=1, 2, \cdots$. Let $V-Q$ be an arbitrary member of $\mathfrak{H}_i$, where $V$ is a member of $\mathfrak{B}_i - \bigcup_\lambda \mathfrak{B}_{\lambda i}$. By the openness of $V$ in $F_i$, there exists an open $V'$ in $X$ such that $V = V' \cap F_i$, and so

$$V = V' \cap F_i = V' \cap \left((X - \bigcup_{j \neq i} F_j) \cup (Q \cap F_i)\right)$$
$$= \left(V' \cap (X - \bigcup_{j \neq i} F_j)\right) \cup (V' \cap Q \cap F_i)$$

and hence $V - Q = V' \cap (X - \bigcup_{j \neq i} F_j) \cap (X - Q)$ is clearly open in $X$.

(2) *$\mathfrak{H}$ is a covering of $X$.* Since $\bigcup_\lambda \mathfrak{B}_\lambda$ is a covering of $\bigcup_\lambda G_\lambda$, let $x_0$ be an arbitrary point of $X - (\bigcup_\lambda \mathfrak{B}_\lambda^{*})$[2] and hence $x_0 \notin \bigcup_\lambda G_\lambda = Q$, and so there exists only one positive integer $i_0$ such that $x_0 \in F_{i_0} - Q$. By the fact that $\mathfrak{B}_{i_0}$ is a covering of $F_{i_0}$, there exists some open set $U_0$ in $X$ such that $x_0 \in U_0 \cap F_{i_0} = V_0 \in \mathfrak{B}_{i_0}$. Since $\operatorname{Int}(V_0) = U_0 \cap \operatorname{Int}(F_{i_0}) \ni x_0$, $x_0 \in \operatorname{Int}(V_0)$ where $V_0 \in \mathfrak{B}_{i_0}$. Accordingly, if $V_0$is a member of $\bigcup_\lambda \mathfrak{B}_{\lambda i_0}$, then $x_0 \in \bigcup_\lambda \mathfrak{B}_\lambda^*$. This is contrary to $x_0 \in X - \bigcup_\lambda \mathfrak{B}_\lambda^*$, and so $x_0 \in V_0 - Q$

2) For the collection $\mathfrak{U}$ of subsets of $X$, $\mathfrak{U}^*$ will denote the set $\cup \{U \mid U \in \mathfrak{U}\}$.

$\in \mathfrak{H}_{i_0}$. This means $x_0 \in \mathfrak{H}^*_{i_0}$.

(3) $\mathfrak{H}$ *is a refinement of* $\mathfrak{W}$. It is obvious that $\mathfrak{H}_i$ is a refinement of $\mathfrak{W}$ for each $i=1, 2, \cdots$ and so let $\lambda_0$ be an arbitrary index of $\Lambda$ and moreover $V_0$ be an arbitrary element of $\mathfrak{L}_{\lambda_0}$. Then we may rewrite as follows:

$$V_0=\mathrm{Int}\left(\bigcup_{i=1}^{n} V^{\lambda_0}_{j(k_i)}\right) \text{ where } V^{\lambda_0}_{j(k_i)} \in \mathfrak{L}_{\lambda_0 k_i} \text{ and } \bigcap_{1}^{n} V^{\lambda_0}_{j(k_i)} \neq \varnothing ,$$

and so there exists a point $x_0$ such that $x_0 \in \bigcap_1^n V^{\lambda_0}_{j(k_i)}$. On the other hand, for each $i=0, 1, \cdots, n$, there exists a member $U_i$ of $\mathfrak{U}$ such that $x_0 \in U_0$, and $x_0 \in V^{\lambda_0}_{j(k_i)} \subset U_i$ for $i=1, 2, \cdots, n$. Therefore $V_0 \subset \bigcup_1^n V^{\lambda_0}_{j(k_i)} \subset \bigcup_1^n U_i \subset st(U_0, \mathfrak{U}) \subset W_{\alpha_0}$ for some $W_{\alpha_0} \in \mathfrak{W}$. This means that $\mathfrak{L}_{\lambda_0}$ is a refinement of $\mathfrak{W}$.

(4) $\mathfrak{H}$ *is star countable.*

(4.1) Let $i_0$ be an arbitrary positive number and $V-Q$ be an arbitrary member of $\mathfrak{H}_{i_0}$ where $V \in \mathfrak{L}_{i_0} - \bigcup_{\lambda} \mathfrak{L}_{\lambda i_0}$. By the definitions of $\{\mathfrak{H}_i \mid i=1, 2, \cdots\}$ and $Q$, $\mathfrak{H}^*_j \cap \mathfrak{H}^*_{i_0} = \varnothing$ for every $j \neq i_0$. If $(V-Q) \cap V_0 \neq \varnothing$ for some $V_0=\mathrm{Int}\left(\bigcup_1^n V^{\lambda}_{j(k_i)}\right) \in \mathfrak{L}_{\lambda}$, where $V^{\lambda}_{j(k_i)} \in \mathfrak{L}_{\lambda k_i}$, then $(V-Q) \cap V^{\lambda}_{j(t)} \neq \varnothing$ for some $t \in \{k_1, k_2, \cdots, k_n\}$. Since $V^{\lambda}_{j(t)} \subset F_t$ and $V-Q \subset \mathrm{Int}(F_{i_0}) = \{y \mid y \notin F_i \text{ for every } i \neq i_0\}$, we have $t=i_0$. This fact shows the following: If $(V-Q) \cap V_0 \neq \varnothing$, then $i_0 \in \{k_1, k_2, \cdots, k_n\}$ and $(V-Q) \cap V^{\lambda}_{j(i_0)} \neq \varnothing$. On the other hand, $\{\lambda \mid V \cap V^{\lambda}_{j(i_0)} \neq \varnothing, V^{\lambda}_{j(i_0)} \in \mathfrak{L}_{i_0}\}$ is countable, and hence $\{\lambda \mid (V-Q) \cap V^{\lambda}_{j(i_0)} \neq \varnothing\}$ is countable by the facts that $\mathfrak{L}_{i_0}$ is star countable and $\{\mathfrak{L}_{\lambda i_0} \mid \lambda\}$ is mutually disjoint. Furthermore $\mathfrak{H}_{i_0}$ is clearly star countable. These mean that $V-Q$ intersects only countably many elements of $\mathfrak{H}$.

(4.2) Let $\lambda_0$ be an arbitrary element of $\Lambda$, and $\mathrm{Int}(V_0)$ be an arbitrary member of $\mathfrak{H}_{\lambda_0}$ where $V_0 = \bigcup \{V^{\lambda_0}_{j(k_i)} \mid V^{\lambda_0}_{j(k_i)} \in \mathfrak{L}_{\lambda_0 k_i}$ for $i=1, 2, \cdots, n\}$. Then, by the definition of $\{\mathfrak{L}_{\lambda k_i} \mid \lambda\}$, all the indices of $\lambda'$ that $V^{\lambda_0}_{k_i}$ intersects $V^{\lambda'}_{k_i}$ is countable for each $i=1, 2, \cdots, n$, and therefore, in order to show that $\mathrm{Int}(V_0)$ intersects only countably many elements of $\bigcup_{\lambda} \mathfrak{L}_{\lambda}$ it is sufficient to show that the set $\{V^{\lambda'}_j \mid V^{\lambda'}_j \in \mathfrak{L}_{\lambda' j}, V^{\lambda_0}_{k_i} \cap V^{\lambda'}_j \neq \varnothing; \lambda \neq \lambda, j \neq k_j\}$ is countable for each $i=1, 2, \cdots, n$. In reality, this set is empty. Lastly we shall show that $\mathrm{Int}(V_0)$ intersects only countably many elements of $\bigcup_1^{\infty} \mathfrak{H}_i$. For this purpose, let $j$ be any integer, then we can consider the two cases: [1] $j \notin \{k_1, k_2, \cdots, k_n\}$ and [2] $j \in \{k_1, k_2, \cdots, k_n\}$. In the first case, $\mathrm{Int}(V_0) \cap \mathfrak{H}^*_j = \varnothing$. In the second case, that is, $j=k_{i_0}$ for some $i_0$ $(1 \leqq i_0 \leqq n)$, "$(V-Q) \cap V_0 \neq \varnothing$" is equivalent to "$(V-Q) \cap V^{\lambda_0}_{k_{i_0}} \neq \varnothing$". Since $\mathfrak{L}_{k_{i_0}}$ is star countable, $V^{\lambda}_{k_{i_0}}$ intersects only countably many elements of $\mathfrak{L}_{k_{i_0}} - \bigcup_{\lambda} \mathfrak{L}_{\lambda k_{i_0}}$ and so $V^{\lambda}_{k_{i_0}}$ intersects only countably many elements of $\mathfrak{H}_{k_{i_0}}$. This shows that $\mathrm{Int}(V_0)$ intersects only countably many elements of $\mathfrak{H}_j$.

From (1), (2), (3), and (4), we can see that $\mathfrak{H}$ is a star countable open refinement of $\mathfrak{W}$. Since $X$ is a regular $T_1$-space, $X$ is strongly paracompact by a theorem of Yu. Smirnov [3].

By use of Theorem 1, we can prove the following main theorem which is also a generalization of V. Trnkova's theorem.

**Theorem 2.** *Let $X$ be a regular $T_1$-space and $\mathfrak{F}=\{F_\alpha \mid \alpha \in A\}$ be a locally finite, star countable closed covering of $X$ such that $\mathfrak{B}(F_\alpha)$ has the locally Lindelöf property for each $\alpha \in A$. Then, in order that the space $X$ be strongly paracompact, it is necessary and sufficient that $F_\alpha$ be a strongly paracompact subspace for each $\alpha \in A$.*

**Proof.** Necessity is obvious and so we shall prove the sufficiency. Let $\{\mathfrak{F}_\lambda \mid \lambda \in \Lambda\}$ be all the components[3] of $\mathfrak{F}$ and $H_\lambda$ be $\mathfrak{F}_\lambda^*$ for each $\lambda \in \Lambda$. Then, by the definition of $\mathfrak{F}$, $H_\lambda$ is open and closed in $X$, and furthermore $\mathfrak{F}_\lambda$ is a countable collection and hence $\{H_\lambda \mid \lambda \in \Lambda\}$ is discrete covering of $X$ such that each $H_\lambda$ is strongly paracompact for each $\lambda \in \Lambda$ by Theorem 1, and so $X$ is strongly paracompact from the mutual disjointedness of $\{H_\lambda \mid \lambda \in \Lambda\}$. This completes the proof.

**§ 2. Applications.** In this section, we shall prove two theorems as the consequences of Theorem 1.

**Definition.** Let $X$ be a topological space and $K$ be a subset of $X$. A space $X$ has *the locally Lindelöf property at $K$* if, for each $x$ of $K$, there exists an arbitrary small neighborhood $U$ of $x$ in $X$ such that $U$ has the Lindelöf property.

**Theorem 3.** *Let $\mathfrak{F}=\{F'_\alpha \mid \alpha \in A\}$ be a locally finite closed covering of a regular $T_1$-space $X$ such that $X$ has the locally Lindelöf property at $\bigcup_{\alpha \in A} \mathfrak{B}(F'_\alpha)$. If $F'_\alpha$ is strongly paracompact for each $\alpha \in A$, then $X$ is strongly paracompact.*

**Proof.** Let $A$ be a well ordered set and $F_1=F'_1$, $F_\alpha=\overline{F'_\alpha-\bigcup_{\beta<\alpha} F'_\beta}$ for every $\alpha>1$. Let $Q=\bigcup_{\alpha\neq\beta}(F_\alpha \cap F_\beta)$, then $Q$ is closed in $X$ and $X$ has the locally Lindelöf property at $Q$ by Lemma 2. Therefore $Q\subset \bigcup_{x\in Q} V(x)$, where $V(x)$ is an open neighborhood of $x$ in $X$ with the Lindelöf closure. It is obvious that $X$ is paracompact and so is normal, and hence there exists an open set $G$ in $X$ such that

3) Let $X$ be a topological space and let $\mathfrak{F}$ be a collection of subsets of $X$. We call that $\mathfrak{F}'$, subcollection of $\mathfrak{F}$, is *connected* if for any two elements $F_\alpha, F_\beta$ of $\mathfrak{F}'$, there exists a finite sequence $F_1, \cdots, F_n$ of $\mathfrak{F}'$ such that $F_1=F_\alpha$, $F_n=F_\beta$ and such that $F_i \cap F_{i+1} \neq \emptyset$ $(1\leqq i\leqq n-1)$. $\mathfrak{F}'$ is called *component* of $\mathfrak{F}$ if no subcollection of $\mathfrak{F}$ which contains $\mathfrak{F}'$ is connected.

$Q \subset G \subset \bar{G} \subset \bigcup_{x \in Q} V(x)$, and so $\bar{G}$ is a neighborhood of $Q$ and a closed paracompact subspace with the locally Lindelöf property. Therefore $\bar{G}$ is strongly paracompact. On the other hand, let $H_\alpha = F_\alpha - G$ for each $\alpha \in A$, then it is easily seen that $\{H_\alpha \mid \alpha \in A\}$ is clearly a discrete closed collection and $H_\alpha$ is strongly paracompact, and so $H = \bigcup_\alpha H_\alpha$ is strongly paracompact closed subspace of $X$. Then $\{H, \bar{G}\}$ is a closed covering of $X$ such that subspaces $H, \bar{G}$ are strongly paracompact and $H \cap \bar{G}$ has the locally Lindelöf property. This implies the strong paracompactness of $X$ by Theorem 1 (or, by V. Trnkova's theorem [5]).

**Theorem 4.** *Let $X$ be a normal $T_1$-space and $\mathfrak{G} = \{G_\alpha \mid \alpha \in A\}$ be a locally finite open covering of $X$. If $G_\alpha$ is a strongly paracompact subspace with the locally Lindelöf property for each $\alpha \in A$, then $X$ is itself strongly paracompact.*

**Proof.** Since $X$ is normal, there exists a closed covering $\{F_\alpha \mid \alpha \in A\}$ of $X$ such that $F_\alpha \subset G_\alpha$ for each $\alpha \in A$ and hence $\{F_\alpha \mid \alpha \in A\}$ is a locally finite closed covering of $X$ such that $F_\alpha$ is strongly paracompact for each $\alpha \in A$. By the assumption, it is easily seen that $X$ has a locally Lindelöf property at $\bigcup_{\alpha \in A} \mathfrak{B}(F_\alpha)$. This completes the proof of Theorem 4.

**Remark.** Theorem 3 is a generalization of Theorem 2 in our previous note [1], from the point of view of obtaining only the strong paracompactness of a space.

In Theorem 5 in the same note [1], we assumed the regularity of $X$ instead of the normality in Theorem 4, and more we assumed the locally Lindelöf property of $\mathfrak{B}(G_\alpha)$ for each $\alpha$. Therefore we may consider that Theorem 4 is a generalization of Theorem 5 in [1].

In conclusion, I express my hearty thanks to Prof. S. Hanai and Mr. A. Okuyama for their kind advices for me.

## References

[1] S. Hanai and Y. Yasui: A note on unions of strongly paracompact spaces. Memoirs of Osaka Gakugei University (to appear).

[2] V. Šedivá: On collectionwise normal and hypocompact spaces. Cech. Math. Jour., **10** (84), 50–61 (1959).

[3] Yu. Smirnov: On strongly paracompact spaces: Izv. Akad. Nauk SSSR, **20**, 253–274 (1959).

[4] A. H. Stone: Paracompactness and product spaces. Bull. Amer. Math. Soc., **54**, 977–982 (1948).

[5] V. Trnkova: Unions of strongly paracompact spaces. Dokl. Akad. Nauk SSSR, **146**, 43–45 (1962), (Soviet Math., **3**, 1248–1250).