29. *An Algebraic Formulation of K-N Propositional Calculus. II*

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In his paper [1], K. Iséki defined the *NK-algebra*. For the details of the *NK-algebra*, see [1]. The conditions of the *NK-algebra* are as follows:

a) $\sim(p*p)*p=0$,

b) $\sim p*(q*p)=0$,

c) $\sim\sim(\sim\sim(p*r)*\sim(r*q))*\sim(\sim q*p)=0$,

d) Let α, β be expressions in this system, then

$$\sim\sim\beta*\sim\alpha=0 \text{ and } \alpha=0 \textit{ imply } \beta=0.$$

In this note, we shall show that a *NK-algebra* is implied by the following conditions:

1) $\sim(p*p)*p=0$,

2) $p*(\sim p*q)=0$,

3) $\sim\sim(\sim\sim(p*r)*\sim(r*q))*\sim(\sim q*p)=0$,

4) $\sim\sim\beta*\sim\alpha=0$ and $\alpha=0$ imply $\beta=0$, where α, β are expressions in this system. We shall prove that 1)–4) imply b)

In 3), put $p=\beta, q=\alpha, r=\gamma$, then

$$\sim\sim(\sim\sim(\beta*\gamma)*\sim(\gamma*\alpha))*\sim(\sim\alpha*\beta)=0.$$

By 4), we have $\sim\sim(\beta*\gamma)*\sim(\gamma*\alpha)=0$. Then we have the followings:

A) $\sim\alpha*\beta=0$ *implies* $\sim\sim(\beta*\gamma)*\sim(\gamma*\alpha)=0$,

B) $\sim\alpha*\beta=0, \gamma*\alpha=0$ *imply* $\beta*\gamma=0$,

C) $\sim\alpha*\beta=0, \sim\gamma*\alpha=0$ *imply* $\beta*\sim\gamma=0$.

In B), put $\alpha=\sim p*\sim p, \beta=\sim p, \gamma=p$, then

$$\sim(\sim p*\sim p)*\sim p=0, \; p*(\sim p*\sim p)=0 \text{ imply } \sim p*p=0.$$

By 1) and 2) we have

5) $\sim p*p=0$.

In 3), put $q=p$, then

$$\sim\sim(\sim\sim(p*r)*\sim(r*p))*\sim(\sim p*p)=0.$$

By 5) we have

6) $\sim\sim(p*r)*\sim(r*p)=0$.

In 3), put $q=\sim p, p=\sim p, r=\sim\sim p$, then

$$\sim\sim(\sim\sim(\sim p*\sim\sim p)*\sim(\sim\sim p*\sim p))*\sim(\sim\sim p*\sim p)=0.$$

By 5), we have

7) $\sim p*\sim\sim p=0$.

In 6), put $p=\alpha, r=\beta$, then $\sim\sim(\alpha*\beta)*\sim(\beta*\alpha)=0$ implies $\alpha*\beta=0$. Hence by 4) we have

D) $\beta*\alpha=0$ *implies* $\alpha*\beta=0$.

By 5) and D) we have

8) $p*\sim p=0$.

In 3), put $p=\sim\sim q, r=\sim r$, and $p=\sim\sim q$, then

$$\sim\sim(\sim\sim(\sim\sim q*\sim r)*\sim(\sim r*q))*\sim(\sim q*\sim\sim q)=0,$$
$$\sim\sim(\sim\sim(\sim\sim q*r)*\sim(r*q))*\sim(\sim q*\sim\sim q)=0.$$

By 8) we have respectively

9) $\sim\sim(\sim\sim q*\sim r)*\sim(\sim r*q)=0$.

10) $\sim\sim(\sim\sim q*r)*\sim(r*q)=0$.

In 10), put $r=\alpha, q=\beta$, then $\sim\sim(\sim\sim\beta*\alpha)*\sim(\alpha*\beta)=0$. Therefore we have

E) $\alpha*\beta=0$ *implies* $\sim\sim\beta*\alpha=0$.

In 9), put $r=\sim\beta, q=\alpha$, then

$$\sim\sim(\sim\sim\alpha*\sim\sim\beta)*\sim(\sim\sim\beta*\alpha)=0.$$

Then by E), we have

F) $\alpha*\beta=0$ *implies* $\sim\sim\alpha*\sim\sim\beta=0$.

In 10), put $r=p, q=\sim p$, then

$$\sim\sim(\sim\sim\sim p*p)*\sim(p*\sim p)=0.$$

By 8) we have

11) $\sim\sim\sim p*p=0$.

In 3), put $p=\sim\beta, q=\sim\alpha, r=\alpha$, then

$$\sim\sim(\sim\sim(\sim\beta*\alpha)*\sim(\alpha*\sim\alpha)*\sim(\sim\sim\alpha*\sim\beta)=0.$$

By 8) $\alpha*\sim\alpha=0$, hence we have

G) $\sim\sim\alpha*\sim\beta=0$ *implies* $\sim\beta*\alpha=0$.

In 3), put $p=\alpha, q=\beta, r=\gamma$, then

$$\sim\sim(\sim\sim(\alpha*\gamma)*\sim(\gamma*\beta))*\sim(\sim\beta*\alpha)=0.$$

And by G), $\sim\sim(\alpha*\gamma)*\sim(\gamma*\beta)=0$ implies $\sim(\gamma*\beta)*(\alpha*\gamma)=0$. Therefore by 4) we have

H) $\sim\beta*\alpha=0$ *implies* $\sim(\gamma*\beta)*(\alpha*\gamma)=0$.

In H), if we put $\beta=\alpha, \alpha=\beta, \gamma=\delta$, and $\beta=\gamma, \alpha=\delta, \gamma=\alpha$, then we have respectively

H_1) $\sim\alpha*\beta=0$ *implies* $\sim(\delta*\alpha)*(\beta*\delta)=0$.

H_2) $\sim\gamma*\delta=0$ *implies* $\sim(\alpha*\gamma)*(\delta*\alpha)=0$.

In C), put $\alpha=\delta*\alpha, \beta=\beta*\delta, \gamma=\alpha*\gamma$, then by H_1) and H_2) we have

I) $\sim\alpha*\beta=0, \sim\gamma*\delta=0$ *imply* $(\beta*\delta)*\sim(\alpha*\gamma)=0$.

In H), put $\alpha=\sim\sim p, \beta=p, \gamma=r$, then by 5) we have

12) $\sim(r*p)*(\sim\sim p*r)=0$.

By 11), put $p=\sim\alpha$, then we have

J) $\alpha=0$ *implies* $\sim\sim\alpha=0$.

In J), put $\alpha=\sim\gamma*\beta$, then we have $\sim\gamma*\beta=0$ implies $\sim\sim(\sim\gamma*\beta)=0$. In 9), put $\gamma=\delta, q=\gamma$, then we have

$\sim\sim(\sim\sim\gamma*\sim\delta)*\sim(\sim\delta*\gamma)=0$ implies $\sim\sim\gamma*\sim\delta=0$.

In I), put $\alpha=\sim\gamma$, $\beta=\sim\delta$, $\gamma=\beta$, $\delta=\alpha$, then we have $\sim\sim\gamma*\sim\delta=0$, $\sim\beta*\alpha=0$ imply $(\sim\delta*\alpha)*\sim(\sim\gamma*\beta)=0$. In F), put $\alpha=\sim\delta*\alpha$, $\beta=\sim(\sim\gamma*\beta)$. then we have $(\sim\delta*\alpha)*\sim(\sim\gamma*\beta)=0$ implies $\sim\sim(\sim\delta*\alpha)*\sim\sim\sim(\sim\gamma*\beta)=0$. Therefore we have the following general theorem.

K) $\sim\beta*\alpha=0$, $\sim\gamma*\beta=0$, $\sim\delta*\gamma=0$ *imply* $\sim\delta*\alpha=0$.

In H), put $\beta=p, \alpha=p, \gamma=r$, then by 5) we have

13) $\sim(r*p)*(p*r)=0$.

In K), put $\delta=\gamma$, then by 5) we have

L) $\sim\beta*\alpha=0$, $\sim\gamma*\beta=0$ *imply* $\sim\gamma*\alpha=0$.

In L), put $\beta=\sim\sim p*q$, $\alpha=\sim\sim q*\sim\sim p$, then by 12) and 2) we have

14) $\sim p*(\sim\sim q*\sim\sim p)=0$.

In H_2), put $\delta=p$, $\gamma=\sim\sim p$, $\alpha=\sim\sim q$, then by 11) we have

15) $\sim(\sim\sim q*\sim\sim p)*(p*\sim\sim q)=0$.

In H_1), put $\alpha=\sim\sim q$, $\beta=q$, $\delta=p$, then by 11) we have

16) $\sim(p*\sim\sim q)*(q*p)=0$.

In K), put $\alpha=q*p$, $\beta=p*\sim\sim q$, $\gamma=\sim\sim q*\sim\sim p$, $\delta=p$, then by 14), 15), and 16) we have

17) $\sim p*(q*p)=0$.

Therefore the proof is complete.

Reference

[1] K. Iséki: An Algebraic Formulation of *K-N* Propositional Calculus. Proc. Japan Acad., **42**, 1164–1167 (1966).