25. On the Crossed Product of Abelian von Neumann Algebras. I

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1. H. A. Dye has successfully investigated in detail the groups of measure preserving transformations on a finite measure space in his deep studies [3] and [4]. Among many others, Dye has introduced the notion "equivalence" among these groups. In the present note, we shall discuss his notion in connection with the crossed product of an abelian von Neumann algebra.

2. Throughout the note, we shall use the terminology on von Neumann algebras due to J. Dixmier [2] without further explanations.

Following after Dye [3], we shall introduce some fundamental definitions on automorphisms of an abelian von Neumann algebra $\mathcal{A}$ with the faithful normal trace ϕ normalized by $\phi(1)=1$. A projection P in $\mathcal{A}$ is said to be *absolutely fixed* under an automorphism g of $\mathcal{A}$ if $Q^g=Q$ for each $Q\leqq P$. For the given two automorphisms g and h of $\mathcal{A}$, we shall denote by $F(g, h)$ the maximal projection in $\mathcal{A}$ which is absolutely fixed under gh^{-1}.

Let G be a group of automorphisms of $\mathcal{A}$ which is ϕ-preserving in the sense that $\phi(A^g)=\phi(A)$ for each $A\in\mathcal{A}$ and $g\in G$. If $F(g, 1)=0$ for each $g\neq 1$ in G, then G is called *freely acting*. If α is an automorphism of $\mathcal{A}$, we say that α *depends* on G if $\text{l.u.b.}_{g\in G} F(\alpha, g)=1$. We shall denote $[G]$ by the collection of all automorphisms of $\mathcal{A}$ which preserve ϕ and depend on G. Two groups of G_1 and G_2 of ϕ-preserving automorphisms of $\mathcal{A}$ will be called *equivalent*, if they determine the same full group, that is, $[G_1]=[G_2]$.

3. At first we shall review briefly the concept of the crossed product of an abelian von Neumann algebra $\mathcal{A}$ with the faithful normal trace ϕ normalized with $\phi(1)=1$ by an enumerable freely acting group G of ϕ-preserving automorphisms of $\mathcal{A}$, cf. for instance [5] and [6].

We shall denote an operator valued function defined on G by $\sum_{g\in G} g\otimes A_g$ where $A_g\in\mathcal{A}$ is the value of the function at $g\in G$. Let $\mathcal{D}$ be the set of all functions such that $A_g=0$ up to a finite subset of G. Then $\mathcal{D}$ is a linear space with the usual operations of the addition and the scalar multiplication, and becomes a *-algebra by the following operations:

$$\Big(\sum_{g\in G} g\otimes A_g\Big)\Big(\sum_{h\in G} h\otimes B_h\Big)=\sum_{g,h\in G} gh\otimes A_g B_h^{g^{-1}}$$

and

$$\Big(\sum_{g\in G} g\otimes A_g\Big)^*=\sum_{g\in G} g^{-1}\otimes A_g^{*g}.$$

For a trace ϕ in $\mathcal{A}$, we can introduce a trace φ in $\mathcal{D}$ by

$$\varphi(g\otimes A_g)=\begin{cases}\phi(A_g) & \text{for } g=1,\\ 0 & \text{for } g\neq 1,\end{cases}$$

and

$$\varphi\Big(\sum_{g\in G} g\otimes A_g\Big)=\sum_{g\in G}\varphi(g\otimes A_g).$$

Then the restriction of φ on $\mathcal{A}=1\otimes\mathcal{A}$ coincides with ϕ and φ is faithful on $\mathcal{D}$, cf. [5]. Let $\mathcal{H}$ be the representation space of $\mathcal{A}$ by ϕ (cf. for instance [2]), then $G\otimes\mathcal{H}$, in the sense of H. Umegaki [7], is the representation space of $\mathcal{D}$ by φ, and $\mathcal{D}$ in represented faithfully on $G\otimes\mathcal{H}$. Put, for each $A\in\mathcal{A}$ and $g\in G$,

$$1\otimes A\Big(\sum_{h\in G} h\otimes B_h\Big)=\sum_{h\in G} h\otimes AB_h \quad\text{and}\quad U_g\Big(\sum_{h\in G} h\otimes B_h\Big)=\sum_{h\in G} gh\otimes B_h^{g^{-1}}\Big)$$

for any $\sum_{h\in G} h\otimes B_h\in\mathcal{D}$, being considered as a dense linear subset of $G\otimes\mathcal{H}$. Then U_g is a unitary operator and we have

$$U_g^*(1\otimes A)U_g=1\otimes A^g.$$

Hereafter, we shall identify $1\otimes\mathcal{A}$ with $\mathcal{A}$ since $\mathcal{A}$ is isomorphic to $1\otimes\mathcal{A}$.

The *crossed product* $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G (with respect to ϕ) is the weak closure of $\mathcal{D}$ on $G\otimes\mathcal{H}$, being considered $\mathcal{D}$ as a *-algebra of operators on $G\otimes\mathcal{H}$.

4. In this section, we shall discuss automorphisms depending on a group in connection with the decomposition of unitary operators obtained in [1].

Let $\mathcal{A}$ and ϕ be as same as in §2, G be a countable freely acting group of ϕ-preserving automorphisms of $\mathcal{A}$ and α be an automorphism of $\mathcal{A}$ which depends on G. Then, by [1; Theorem 1], there exists a unitary operator V_α in the crossed product $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G (with respect to ϕ) such that

$$V_\alpha=\sum_{g\in G} E_g U_g \quad\text{and}\quad V_\alpha^* A V_\alpha=A^\alpha,$$

for any $A\in\mathcal{A}$, where E_g satisfies the following properties:

(1) *E_g is a projection in $\mathcal{A}$ for every $g\in G$,*

(2) *$E_gE_h=0$ for $g\neq h$,*

(3) $\sum_{g\in G} E_g=1$,

(4) *E_g is absolutely fixed under αg^{-1}.*

Furthermore, E_g satisfies the following

Lemma 1. *$E_g = F(\alpha, g)$ for any $g \in G$.*

Proof. We have at once $E_g \leqq F(\alpha, g)$ by (4). If $E_g \neq F(\alpha, g)$, then $F(\alpha, g)E_h \neq 0$ for some $h \neq g$ by (3). For each projection Q in $\mathcal{A}$ such that $Q \leqq F(\alpha, g)E_h$, we have $Q^g = Q^\alpha = Q^h$. So $F(\alpha, g)E_h$ is absolutely fixed under gh^{-1}, which is a contradiction. Therefore, $E_g = F(\alpha, g)$ as desired.

For V_α, we have the following lemma:

Lemma 2. *Let α and β be automorphisms of $\mathcal{A}$ which depend on G, then*

$$V_\alpha V_\beta = V_{\alpha\beta} \quad \text{and} \quad V_\alpha^* = V_\alpha^{-1}.$$

That is, $\alpha \to V_\alpha$, is a representation of $[G]$.

Proof. By Lemma 1, we have

$$V_\alpha = \sum_{g \in G} F(\alpha, g) U_g \quad \text{and} \quad V_\beta = \sum_{g \in G} F(\beta, g) U_g.$$

Hence

$$(5) \qquad \begin{aligned} V_\alpha V_\beta &= \sum_{g,h \in G} F(\alpha, g) F(\beta, h)^{g^{-1}} U_{gh} \\ &= \sum_{g_0 \in G} \Big(\sum_{gh = g_0} F(\alpha, g) F(\beta, h)^{g^{-1}} U_{g_0} \Big). \end{aligned}$$

Now, we shall show

$$\sum_{g \in G} F(\alpha, g) F(\beta, g^{-1}g_0)^{g^{-1}} = F(\alpha\beta, g_0)$$

for any $g_0 \in G$. For every projection Q which satisfies

$$Q \leqq F(\alpha, g) F(\beta, g^{-1}g_0)^{g^{-1}},$$

we have $Q^\alpha = Q^g$ and $Q^{g\beta} = Q^{g_0}$, whence $Q^{\alpha\beta} = Q^{g\beta} = Q^{g_0}$. Hence

$$F(\alpha, g) F(\beta, g^{-1}g_0)^{g^{-1}} \leqq F(\alpha\beta, g_0)$$

for any $g, g_0 \in G$. On the other hand, since G is freely acting, we have

$$[F(\alpha, g_1) F(\beta, g_1^{-1}g_0)^{g_1^{-1}}][F(\alpha, g_2) F(\beta, g_2^{-1}g_0)^{g_2^{-1}}] = 0,$$

if $g_1 \neq g_2$. Therefore, we have

$$\sum_{g \in G} F(\alpha, g) F(\beta, g^{-1}g_0)^{g^{-1}} \leqq F(\alpha\beta, g_0),$$

for any $g_0 \in G$. If

$$P = F(\alpha\beta, g) - \sum_{g \in G} F(\alpha, g) F(\beta, g^{-1}g_0) g^{-1} \neq 0,$$

for some $g_0 \in G$. Then there exists $g_1 \in G$ such that $F(\alpha, g_1)P \neq 0$ since $\sum_{g \in G} F(\alpha, g) = 1$, and there exists $g_2 \in G$ such that $F(\beta, g_2)^{g_1^{-1}} F(\alpha, g_1)P \neq 0$ by

$$\sum_{g \in G} F(\beta, g)^{g_1^{-1}} = \Big[\sum_{g \in G} F(\beta, g)\Big]^{g_1^{-1}} = 1.$$

Furthermore, for any projection dominated by $F(\beta, g_2)^{g_1^{-1}} F(\alpha, g_1)P$, we have $Q^{g_1 g_2} = Q^{g_1\beta}$, $Q^\alpha = Q^{g_1}$ and $Q^{\alpha\beta} = Q^{g_0}$. Hence, we have $Q^{g_0} = Q^{g_1 g_2}$, that is, the nonzero projection $F(\beta, g_2)^{g^{-1}} F(\alpha, g_1)P$ is absolutely fixed under $g_0(g_1 g_2)^{-1}$. By the assumption that G is freely acting, we have $g_0 = g_1 g_2$, whence we have

$$F(\alpha, g_1)F(\beta, g_1^{-1}g_0)^{g_1^{-1}}P \neq 0,$$

which contradicts to the definition of P. Therefore $P=0$ and so the assertion is proved. Thus, we have by (5)

$$V_\alpha V_\beta = \sum_{g_0 \in G} F(\alpha\beta, g_0)U_{g_0} = V_{\alpha\beta}.$$

The remainder half of the lemma is a consequence of a direct calculation:

$$V_\alpha^* = \sum_{g \in G} U_g^* F(\alpha, g) = \sum_{g \in G} F(\alpha, g)^g U_{g^{-1}} = \sum_{g \in G} F(\alpha^{-1}, g)U_g = V_{\alpha^{-1}}.$$

5. We shall give a characterization of the equivalence among groups of automorphisms of an abelian von Neumann algebra in terms of the crossed product:

Theorem 1. *Let $\mathcal{A}$ be an abelian von Neumann algebra with a normalized faithful normal trace ϕ, and let G_1 and G_2 be two countable freely acting groups of ϕ-preserving automorphisms of $\mathcal{A}$. The necessary and sufficient condition that G_1 and G_2 are equivalent is that there exists an isomorphism Φ of the $G_1 \otimes \mathcal{A}$ onto $G_2 \otimes \mathcal{A}$ which preserves $\mathcal{A}$ in elementwise in the sense $\Phi(A)=A$ for any $A \in \mathcal{A}$.*

Proof. If G_1 and G_2 are equivalent, by the definition $[G_1]=[G_2]$. Then $G_1 \subset [G_1]$ and $G_2 \subset [G_1]$. Therefore, for any $g \in G_1$, the unitary operator $V_g = \sum_{h \in G_2} F(g, h)U_h$ of $G_2 \otimes \mathcal{A}$ is corresponding. Using V_g, we can define a linear mapping U' on a dense set $\mathcal{D}_1$ of $G_1 \otimes \mathcal{H}$ into $G_2 \otimes \mathcal{H}$ by $U'(g \otimes A) = AV_g$ and $U'\left(\sum_{i=1}^n g_i \otimes A_i\right) = \sum_{i=1}^n U'(g_i \otimes A_i)$, for $g, g_i \in G_1$, $A, A_i \in \mathcal{A}$, where each element of $\mathcal{A}$ is considered as an element of the canonical representation space $\mathcal{H}$ defined by ϕ. The following computation shows that U' is an isometry:

$$\begin{aligned}
\left\| U'\left(\sum_{i=1}^n g_i \otimes A_i\right) \right\|_2^2 &= \left\| \sum_{i=1}^n U'(g_i \otimes A_i) \right\|_2^2 \\
&= \left\| \sum_{i=1}^n A_i V_{g_i} \right\|_2^2 \\
&= \left\| \sum_{i=1}^n \sum_{g \in G_2} g \otimes A_i F(g_i, g) \right\|_2^2 \\
&= \sum_{g \in G_2} \phi\left[\sum_{i=1}^n A_i^* A_j F(g_i, g) F(g_i g) \right] \\
&= \sum_{g \in G_2} \phi\left[\sum_{i=1}^n A_i^* A_i F(g_i, g) \right] \\
&= \sum_{i=1}^n \phi(A_i^* A_i) \\
&= \left\| \sum_{i=1}^n g_i \otimes A_i \right\|_2^2,
\end{aligned}$$

(because by the assumption that G is freely acting, $F(g_i, g)$ is orthogonal to $F(g_j, g)$ if $i \neq j$). Therefore U' can be extended to an isometric linear mapping U on $G_1 \otimes \mathcal{H}$ into $G_2 \otimes \mathcal{H}$.

Similarly, for $h \in G_2$, there is a unitary operator in $G_1 \otimes \mathcal{A}$ such that

$$V_h = \sum_{g \in G_1} F(g, h) U_g \ .$$

Putting

$$V'(h \otimes A) = AV_h \quad \text{and} \quad V'\Big(\sum_{i=1}^{n} h_i \otimes A_i\Big) = \sum_{i=1}^{n} V'(h_i \otimes A_i),$$

we have an isometric linear mapping V' on the dense set $\mathcal{D}_2$ of $G_2 \otimes \mathcal{H}$ into $G_1 \otimes \mathcal{H}$. Hence, again, we have the extension V of V' which maps isometrically $G_2 \otimes \mathcal{H}$ into $G_1 \otimes \mathcal{H}$.

For these isometric transformations, we have

$$\begin{aligned} VU(g_0 \otimes A) &= V\Big[\sum_{h \in G_2} h \otimes AF(g_0, h)\Big] \\ &= \sum_{g \in G_1, h \in G_2} g \otimes AF(g_0, h)F(g, h) \\ &= \sum_{h \in G_2} g_0 \otimes AF(g_0, h) \\ &= g_0 \otimes A, \end{aligned}$$

for any $g_0 \in G_1$ and $A \in \mathcal{A}$. Hence $VU = 1$ on $G_1 \otimes \mathcal{H}$. Similarly $UV = 1$ on $G_2 \otimes \mathcal{H}$. Therefore, U is an isometric transformation of $G_1 \otimes \mathcal{H}$ onto $G_2 \otimes \mathcal{H}$ and V is its inverse.

In the next place, for $\sum_{i=1}^{n} A_i U_{g_i} \in G_1 \otimes \mathcal{A}$, we define a linear mapping Φ' of $G_1 \otimes \mathcal{A}$ into $G_2 \otimes \mathcal{A}$ by

$$\Phi'\Big(\sum_{i=1}^{n} A_i U_{g_i}\Big) = \sum_{i=1}^{n} A_i V_{g_i},$$

where

$$V_{g_i} = \sum_{h \in G_2} F(g_i, h) U_h.$$

Then by Lemma 2 Φ' satisfies

$$\Phi'[(AU_g)^*] = [\Phi'(AU_g)]^*$$

and

$$\Phi'[(AU_g)(BU_h)] = \Phi'(AU_g)\Phi'(BU_h),$$

for $A \in \mathcal{A}$ and $g, h \in G_1$. Furthermore, for $B \in \mathcal{A}$, $g \in G_1$, and $h \in G_2$,

$$UAU_gU^{-1}(h \otimes B) = \sum_{k \in G_1, k' \in G_2} k' \otimes A(F(k, h)B)g^{-1}F(gk, k').$$

On the other hand, by the proof of Lemma 2, we have

$$F(gh, k') = \sum_{k \in G_1} F(k, h)^{g^{-1}} F(gk, k').$$

Hence we have

$$\begin{aligned} UAU_gU^{-1}(h \otimes B) &= \sum_{k' \in G_2} k' \otimes AB^{g^{-1}} F(gh, k') \\ &= \sum_{k' \in G_2} k'h \otimes AB^{g^{-1}} F(gh, k'h) \\ &= \sum_{k' \in G_2} k'h \otimes A(BF(g, k')^g)^{g^{-1}} \\ &= \sum_{k' \in G_2} k'h \otimes A(BF(g^{-1}, k'^{-1}))^{g^{-1}} \\ &= \sum_{k' \in G_2} k'h \otimes AB^{k'^{-1}} F(g, k') \\ &= AV_g(h \otimes B) = \Phi'(AU_g)(h \otimes B). \end{aligned}$$

Therefore, Φ' can be extended to a spatial isomorphism Φ of $G_1 \otimes \mathcal{A}$ onto $G_2 \otimes \mathcal{A}$, and Φ satisfies the required $\Phi(A)=A$ for any $A \in \mathcal{A}$ by the definition of Φ'.

Conversely, if there exists such an isomorphism Φ, then it is automatically spatial on $G_1 \otimes \mathcal{H}$ onto $G_2 \otimes \mathcal{H}$. Since by [1; Theorem 1] there exists a unitary operator U_α in $G_1 \otimes \mathcal{A}$ for any $\alpha \in [G_1]$, $\Phi(U_\alpha)$ is a unitary operator in $G_2 \otimes \mathcal{A}$. $\Phi(U_\alpha)$ induces an inner automorphism which coincides with α on $\mathcal{A}$ by

$$\Phi(U_\alpha)^* A \Phi(U_\alpha) = \Phi(U_\alpha^* A U_\alpha) = \Phi(A^\alpha) = A^\alpha, \qquad \text{for } A \in \mathcal{A}.$$

Hence, by [1; Theorem 2], we have that α depends on G_2, that is, $[G_1] \subset [G_2]$. Symmetrically, we have also $[G_2] \subset [G_1]$. This completes the proof of Theorem 1.

An application of Theorem 1 will be discussed in the next note with the characterization of "weak equivalence" of Dye in terms of the crossed product of abelian von Neumann algebras.

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