24. On a Generalization of a Theorem of Cox

By Masahiro NAKAMURA*) and Midori YOSHIDA**)

(Comm. by Kinjirô KUNUGI, M.J.A., Feb. 13, 1967)

1. If T is a real-valued continuous function defined on a compact Hausdorff space X which satisfies

(1) $$\|T^n - I\| \leqq d < 1, \qquad n=0, 1, 2, \cdots$$

If $|T^n(x)|>1$ for some x, then $|T^n(x)|\to+\infty$ as $n\to\infty$ and so contradicts to (1). If $|T^n(x)|<1$ for some x, then $|T^n(x)|\to 0$ as $n\to\infty$ and contradicts (1) too. Therefore, (1) implies

(2) $$T=I.$$

Similar proof is also possible for complex-valued case.

If A is a B^*-algebra which is commutative and contains the identity, then the Gelfand-Neumark representation theorem gives that A is isometrically isomorphic to the algebra of all continuous complex-valued functions defined on a compact Hausdorff space X which is homeomorphic to the spectrum of A, see for instance [2]. Hence, by the above argument, (1) *implies* (2) *for any element* T *of a commutative* B^**-algebra* A.

2. An analogous proof is also valid for a commutative semisimple Banach algebra, since the Gelfand representation gives $|T(x)|\leqq\|T\|$ for any maximal ideal x. However, the above argument is unable to trace for a commutative Banach algebra having non-zero radical, since $T(x)=1$ for all maximal ideals does not insure (2).

The existence of the non-zero radical is not sufficient to deny the statement that (1) implies (2). For example, if T satisfies (1) and

$$T=I+R, \qquad R^2=0,$$

then the algebra generated by $\{T^n; n=0, 1, 2, \cdots\}$ contains the non-zero radical which contains at least R. In this case (2) is true since otherwise the sphere of radius d centered at the identity contains an unbounded set $\{I+nR; n=0, 1, 2, \cdots\}$.

3. Very recently, R. H. Cox [1] announces without proof that (1) *implies* (2) *if* T *is a square matrix of finite order being considered as a linear operator defined on a finite dimensional euclidean space.* This is an advance towards the problem under the existence of the radical, since the algebra generated by an arbitrary matrix is not necessarily semisimple.

*) Osaka Gakugei Daigaku.
**) Osaka Kentiku Zimusyo.

In the present note, the theorem of Cox will be generalized for an infinite dimensional space as follows:

Theorem. *If T is a (bounded) linear operator defined on a Hilbert space H which satisfies* (1), *then T is the identity operator.*

4. The following proof is based on the mean ergodic theorem, see for instance [3]: *Let $\{T^n; n=0, 1, 2, \cdots\}$ be a semigroup which is bounded in the sense that there exists a positive number M such as $\| T^n \| \leqq M$ for $n=0, 1, 2, \cdots$. Then, for any vector x of H, the mean*

$$x_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

of its iterates converges metrically to a fix point a. Since (1) implies $\| T^n \| = \| I-(I-T^n) \| \leqq \| I \| + \| I-T^n \| < 2$ for $n=0, 1, 2, \cdots$, the mean ergodic theorem is now applicable.

Let F be the set of all fix points under T. It is obvious that F is the proper space of T belonging to the proper value 1, whence F is a (closed) subspace of H. If F coincides with H, then every vector of H is invariant under T, and so $T=I$ which is desired.

Now, let us assume that F is properly contained in H. Let K be the convex closure of all iterates $\{T^k x; k=0, 1, 2, \cdots\}$. If x is a vector of norm unity, then (1) implies that K lies in the sphere of radius d centered at x. On the other hand, if x is a vector which is orthogonal to F, then the distance from x to F is 1, whence $K \cap F$ is void by the hypothesis. This contradicts to the mean ergodic theorem which asserts that $K \cap F$ contains a. The theorem is now proved.

5. Since the mean ergodic theorem depends on the weak compactness of the spheres of the given space, the above proof is valid if the Hilbert space H is replaced by a reflexive Banach space with a minor modification. The theorem remains true for a Banach space if T is weakly almost periodic in the sense that the set of all iterates of an arbitrary vector is always weakly compact. The weak complete continuity is thus sufficient for the theorem. However, this is not a substantial extension: If T is weakly completely continuous, then $I=TT^{-1}$ is weakly completely continuous by the fact that T is invertible by (1), which is equivalent to assume that the space is reflexive.

References

[1] R. H. Cox: Matrices all of whose power lie close the identity (Abstract). Amer. Math. Monthly, **73**, 813 (1966).
[2] C. E. Rickart: General theory of Banach algebras. D. van Nostrand, Princeton (1960).
[3] K. Yosida: Mean ergodic theorem in Banach spaces. Proc. Japan Acad., **14**, 292-294 (1938).