20. *A Proof for the Imbedding Theorems for Sobolev Spaces*

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The purpose of the present paper is to show that we can give another proof for the imbedding theorems for Sobolev spaces, without making use of the customary estimate of multi-dimensional potentials ([1]-[3], [6]-[9]).

Theorem. *Let Ω be a domain in the n-dimensional Euclidean space $\boldsymbol{R}^n$. Assume that there exists a constant R, a cube Q and an open covering $\{\Omega_j\}$ of Ω such that the diameter of Ω_j is not greater than R, and Ω_j is star-shaped with respect to a cube Q_j congruent to Q. In cases $m>0$, assume that for each point x in Ω the number of Ω_j containing x is not greater than a constant N. Then there holds the imbedding*:

$$W^{l,p}(\Omega)\longrightarrow W^{k,q,n-m}(\Omega)$$

if $l-\frac{n}{p}+\frac{m}{q}-k\geqq 0$, $1<p\leqq q<\infty$ *and if either one of the conditions*

(i) $p<q, m>0$, (ii) k *is not an integer*, $m=0$, *and* (iii) $m=n$, *is satisfied.*

For functions $f(x)$ on Ω, we define

$$\|f\|_{L^{p,n-m}(\Omega)}=\sup_{x^{(m)}}\|f(x_1,\cdots,x_m,x^{(m)})\|_{L^p(\Omega_{x^{(m)}})},$$

where $x^{(m)}=(x_{m+1},\cdots,x_n)$ and $\Omega_{x^{(m)}}$ is the set of points $(x_1,\cdots,x_m)$ such that $(x_1,\cdots,x_m,x^{(m)})\in\Omega$, and for $f\in C^\infty(\Omega)$

$$\|f\|_{W^{l,p,n-m}(\Omega)}=\|f\|_{L^{p,n-m}(\Omega)}+\sum_{|\alpha|=l}\|f^{(\alpha)}\|_{L^{p,n-m}(\Omega)}\quad (l \text{ is an integer.})^{*)}$$

or

$$\|f\|_{W^{l,p,n-m}(\Omega)}=\|f\|_{W^{[l],p,n-m}(\Omega)}+\sum_{|\alpha|=[l]}\left\|\frac{f^{(\alpha)}(x)-f^{(\alpha)}(y)}{|x-y|^{(m/p)+\sigma}}\right\|_{L^{p,2n-2m}(\Omega\times\Omega)}$$

$$(l=[l]+\sigma,\ 0<\sigma<1)$$

Here the *spaces* $W^{l,p,n-m}(\Omega)$ *are defined* as the completions of subsets of $C^\infty(\Omega)$ consisting of functions f with $\|f\|_{W^{l,p,n-m}}<\infty$. $W^{l,p}=W^{l,p,0}$, $C^l=W^{l,p,n}$.

In the following we assume that Ω is bounded and is starshaped with respect to a cube. It is easy to extend the results for general domains.

Let Ω be a bounded star-shaped domain with respect to a cube

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*) $f^{(\alpha)}(x)$ denotes the α-th derivative of $f(x)$.

Q and choose $\varphi(x)\in C^\infty(R^n)$ such that $\int\varphi(x)dx=1$ and $\varphi(x)$ is identically equal to zero outside Q. For $f(x)\in C^\infty(\Omega)$, we have by Taylor's formula,

$$f(x)=\sum_{|\alpha|<l}\frac{(x-z)^\alpha}{\alpha!}f^{(\alpha)}(z)+l\sum_{|\alpha|=l}\int_0^1\frac{(x-z)^\alpha}{\alpha!}f^{(\alpha)}(x+t(z-x))t^{l-1}dt.$$

Multiplying by $\varphi(z)$ and integrating with respect to z, we have

$$f(x)=\sum_{|\alpha|=l}\int_0^1 t^{l-1}dt\int\frac{l}{\alpha!}(x-z)^\alpha\varphi(z)f^{(\alpha)}(x+t)(z-x))dz$$

$$+\sum_{|\alpha|<l}\int\varphi_\alpha(z)(x-z)^\alpha f(z)dz, \qquad (1)$$

where $\varphi_\alpha(z)$ is a linear combination of derivatives of $\varphi(z)$. In the same way, we have

$$f(x)=\int_0^1\frac{dt}{t}\iint\psi(x,z)\varphi(w)\{f(x+t(z-x))-f(x+t(w-x))\}dzdw$$

$$+\int\varphi(z)f(z)dz, \qquad (2)$$

where $\psi(x,z)=n\varphi(z)+\sum_{j=1}^n(z_j-x_j)\frac{\partial\varphi}{\partial z_j}(z)$.

Our proof of the Theorem is based on these formulas and the following

Lemma 1. *Let Ω be bounded and star-shaped with respect to Q. Set $w=\Omega\cap S$ where S is a $(n-m)$-dimensional subspace of $\boldsymbol{R}^n$ and let $K(x,z)$ $(x\in S, z\in\boldsymbol{R}^n)$ be a C^h-function having support contained in $S\times Q$. For any function f such that for any compact subset B in w,*

$$\sup_{x\in B}\sup_{0<t\leqq 1}\int_Q|f(x+t(z-x))|dz<\infty$$

(for convenience, we will say that $f(x)$ is of the class $K(\omega)$ if $f(x)$ has this property), we define

$$F(x)=\int_0^T\eta(t)dt\int f(x+t(z-x))K(x,z)dz$$

where $0\leqq|\eta(t)|\leqq t^{l-1}$, $0\leqq T\leqq 1$, and $l>0$. Assume that $\lambda=l-\frac{n}{p}+\frac{r}{q}-k\geqq 0$, $h>k$, $1<p\leqq q<\infty$ and assume further one of the following conditions:

(i) *$p<q$, $r>0$, and* (ii) *k is not an integer, $k<l$, $r=0$,* (iii) *$\lambda>0$. Then*

$$\|F(x)\|_{W^{k,q,n-m-r}(\omega)}\leqq CT^\lambda\|f\|_{L^p(\Omega)} \qquad (3)$$

holds for any $f\in L^p(\Omega)\cap K(\omega)$, where C is a constant independent of f and T.

Proof of the Theorem. Note that the theorem is proved by showing the estimate

$$\|f\|_{W^{k,q,n-m}(\Omega)}\leqq C\|f\|_{W^{l,p}(\Omega)} \qquad \text{(foy any } f\in C^\infty(\Omega)\cap W^{l,p}C\Omega).$$

If l is an integer, then we have this estimate by (1) and Lemma 1. Next consider the case in which $0 \leqq k < l < 1$. The first term on the right side of (2) can be written in the form

$$F(x)=\int_0^1 t^{\mu-1}dt\iint \tilde{f}(x+t(z-x),\, x+t(w-x))K(x, z, w)dzdw,$$

where $\tilde{f}(x, y)=\{f(x)-f(y)\}\,|\,x-y\,|^{-\mu}$, $K(x, z, w)=\psi(x, z)\varphi(w)\,|\,z-w\,|^{\mu}$, $\mu=l+\dfrac{n}{p}$. By Lemma 1, choosing $S=\{(x, x)\,|\,x\in \boldsymbol{R}^n\}$, we have

$$\|F\|_{W^{k,q,n-m}(\Omega)}\leqq C\,\|\tilde{f}\|_{L^p(\Omega\times\Omega)}.$$

Thus the theorem is proved in this case. Consider now the case in which $l=[l]+\sigma$, $0<\sigma<1$, $k\geqq[l]$. By the result just proved we have $\|f^{(\alpha)}\|_{W^{k,q,n-m}}\leqq C\,\|f^{(\alpha)}\|_{W^{l,p}}$ for any α with $|\alpha|=[l]$. Therefore, the theorem is proved in this case also. Choosing a suitable r, and considering two imbeddings $W^{l,p}\to W^{[l],r}$ and $W^{[l],r}\to W^{k,q,n-m}$, we can prove the theorem in general cases.

To prove Lemma 1 we will use the following.

Lemma 2. *Let Ω, ω, $\eta(t)$, Q, T be as in Lemma* 1. *Assume that* (i) $\mu=l-\dfrac{n}{p}+\dfrac{r}{q}\geqq 0$, $p<q$, $r>0$, *or* (ii) $\mu>0$. *Then for any function* $f\in L^p(\Omega)\cap K(\omega)$

$$\left\|\int_0^T \eta(t)dt\int_Q |\,f(x+t(z-x))\,|\,dz\right\|_{L^{q,n-m-r}(\omega)}\leqq CT^{\mu}\,\|f\|_{L^p(\Omega)}. \tag{4}$$

Proof. We may suppose that $S=\{(x', 0)\,|\,x'\in \boldsymbol{R}^{n-m}\}$. By Hölder's inequality we have

$$\int_Q |\,f(x+t(z-x))\,|\,dz\leqq C_1\int_{Q'}dz'\Big(\int_{Q_{x'+t(z'-x')}}|\,f(x'+t(z'-x'),\, tz'')\,|^p dz''\Big)^{1/p}$$
$$\leqq C_1 t^{-\frac{m}{p}}\int_{Q'} g(x'+t(z'-x'))dz',$$

where $Q'=S\cap Q$, $Q_{x'}=\{x''\,|\,(x', x'')\in Q\}$, $\Omega_{x'}=\{x''\,|\,(x', x'')\in\Omega\}$ and $g(x')=\|f(x', x'')\|_{L^p(\Omega_{x'})}$.

Thus it is sufficient to consider the case where $m=0$ and $\omega=\Omega$.

Case (i). We may suppose that $\lambda=0$, $T=1$, and $\eta(t)=t^{l-1}$. By Hölder's inequality we have

$$\int_Q |\,f(x+t(z-x))\,|\,dz\leqq C_1\int_{Q'}dz'\Big(\int_{Q_{z'}}|\,f(x+t(z-x))\,|^p dz''\Big)^{1/p}$$
$$\leqq C_1 t^{-\frac{n-r}{p}}\int_{Q'} g(x'+t(z'-x'))dz',$$

where $z=(z', z'')$, $z'\in\boldsymbol{R}^r$, $z''\in\boldsymbol{R}^{n-r}$, and $g(x')=\|f(x', x'')\|_{L^p(\Omega_{x'})}$. Therefore, it is sufficient to consider the case $r=n$. Put $u=x+t(z-x)$, $s=|\,x-z\,|$, then we have

$$\int_0^1 t^{l-1}dt\int_Q |\,f(x+t(z-x))\,|\,dz\leqq C_2\int_0^R\int_\Omega |\,f(u)\,|\,|\,u-x\,|^{-n+l}s^{n-l-1}duds$$
$$\leqq C_3\int_\Omega |\,f(u)\,|\,|\,u-x\,|^{l-n}du.$$

Applying Riesz's inequality ([4], [5]) to the right side of this inequality, we have the estimate (4).

Case (ii). It is sufficient to consider two cases; (a) $r=n$, $p=q$, and (b) $r=0$. By Hölder's inequality we have

$$\int_0^T \eta(t)dt\int_Q |f(x+t(z-x))|\,dz \leqq C_1\int_0^T t^{l-1}dt\Big(\int_Q |f(x+t(z-x))|^p dz\Big)^{1/p}$$
$$\leqq C_1\int_0^T t^{l-\frac{n}{p}-1}dt\,\|f\|_{L^p(\Omega)},$$

so that the estimate is verified in case (b). Finally consider case (a). By virtue of Hölder's inequality and Jessen's inequality, we have

$$\Big\|\int_0^T \eta(t)dt\int_Q |f(x+t(z-x))dz|\Big\|_{L^p(\Omega)}$$
$$\leqq C_1\int_0^T t^{l-1}dt\Big(\int_\Omega\int_Q |f(x+t(z-x))|^p\,dzdx\Big)^{1/p}$$

Set $u=x+t(x-z)$, $v=x-z$. Then we have

$$\int_\Omega\int_Q |f(x+t(z-x))|^p\,dxdz \leqq \int_\Omega\int_{\Omega-Q} |f(u)|^p\,dudv \leqq C_2\,\|f\|^p_{L^p(\Omega)}.$$

Thus we have the estimate (4) in this case.

Proof of Lemma 1. We may suppose that $S=\{(\hat{x},0)\,|\,\hat{x}\in R^{n-m}\}$ and $\eta(t)=t^{l-1}$. The estimate $\|F\|_{L^{q,n-m-r}(\omega)}\leqq CT^{\mu}\|f\|_{L^p(\Omega)}$ is an immediate consequence of Lemma 2, so that Lemma 1 is proved when $k=0$.

Case $0<k<1$, $r>0$. Set

$$F(x,y)=\int_{t\leqq|x-y|}\Phi(t,x)t^{l-1}dt,\quad \Phi(t,x)=\int_Q |f(x+t(z-x))|\,dz,$$
$$G(x,y)=\int_{t\geqq|x-y|}t^{l-1}dt\left|\int\{f(x+t(z-x))K(x,z)-f(y+t(z-y))K(y,z)\}dz\right|.$$

Then we have

$$|F(x)-F(y)|\leqq C_0\{F(x,y)+F(y,x)\}+G(x,y). \tag{5}$$

It is easily checked that

$$F(x,y)\leqq|x-y|^k G(x),\quad G(x)=\int_0^T t^{l-k-1}\Phi(t,x)dt.$$

Thus we have

$$\int_{\omega(y'')}\frac{F(x,y)^q}{|x-y|^{r+kq}}dy'\leqq C_2G(x)^{q-1}\int_{t\leqq|x-y|}\int\frac{\Phi(t,x)t^{l-1}}{|x-y|^{r+k}}dtdy'\leqq C_3G(x)^q$$

where $y=(y',y'')$, $y'\in \boldsymbol{R}^r$, and $\omega(y'')=\{y'\,|\,(y',y'')\in\omega\}$. It follows from this estimate and Lemma 2 that

$$\left\|\frac{F(x,y)}{|x-y|^{(r/q)+k}}\right\|_{L^{q,2(n-m-r)}(\omega\times\omega)}\leqq CT^{\lambda}\|f\|_{L^p(\Omega)}. \tag{6}$$

Now consider $G(x,y)$. By the identity

$$\int f(y+t(z-y))K(y,z)dz=\int f(x+t(z-x))K\Big(y,z+\frac{1-t}{t}(x-y)\Big)dz$$

and the inequality

$$\left|K(x, z)-K\left(y, z+\frac{1-t}{t}(x-y)\right)\right| \leqq M\frac{|x-y|^{k+\varepsilon}}{t^{k+\varepsilon}},$$

and $0<\varepsilon<1-k, h-k$, we have $G(x, y) \leqq C_1 H(x, y)+H(y, x)$ where

$$H(x, y)=\int_{t \geqq |x-y|} t^{l-k-1}|x-y|^{k+\varepsilon}\Phi(t, x)dt.$$

By Jessen's inequality we have

$$\left(\int_{\omega(y'')} \frac{H(x, y)^q}{|x-y|^{r+kq}} dy'\right)^{1/q}$$

$$\leqq \int_0^T t^{l-k-\varepsilon-1}\Phi(t, x)\left(\int_{|x-y| \leqq t} |x-y|^{\varepsilon q-r} dy'\right)^{1/q} dt \leqq C_2 G(x).$$

Therefore, by Lemma 2 we have

$$\left\| \frac{G(x, y)}{|x-y|^{(r/q)+k}} \right\|_{L^{q,2(n-m-r)}(\omega \times \omega)} \leqq C T^\lambda \|f\|_{L^p(\Omega)}. \tag{7}$$

It follows from (5), (6), and (7) that $\|F(x)\|_{W^{k,q,n-m-r}(\omega)} \leqq C T^\lambda \|f\|_{L^p(\Omega)}$.

Case $0<k<1$, $r=0$. We may assume that $l-\dfrac{n}{p}=k$. By Hölder's inequality we have

$$F(x, y) \leqq C_1 \int_{t \leqq |x-y|} t^{k-1} dt \, \|f\|_{L^p(\Omega)} \leqq C_2 |x-y|^k \|f\|_{L^p(\Omega)} \tag{8}$$

and

$$G(x, y) \leqq C_1 \int_{t \geqq |x-y|} |x-y|^{k+\varepsilon} T^{-\varepsilon-1} dt \, \|f\|_{L^p(\Omega)} \leqq C_3 |x-y|^k \|f\|_{L^p(\Omega)}. \tag{9}$$

By (5), (8), and (9) we have $\dfrac{|F(x)-F(y)|}{|x-y|^k} \leqq C\|f\|_{L^p(\Omega)}$.

Case $k \geqq 1$. Set

$$F_\varepsilon(x)=\int_\varepsilon^T \eta(t)dt \int f(x+t(z-x))K(x, z)dz$$
$$=\int_\varepsilon^T \eta(t)t^{-n}dt \int f(u)K\left(x, x+\frac{u-x}{t}\right)du.$$

Letting $\varepsilon \to 0$ after differentiating and changing the variables of integration we have

$$D_x^\alpha F(x)=\sum_{\alpha \geqq \beta} \binom{\alpha}{\beta} \int_0^T \eta_\beta(t)dt \int f(x+t(z-x))K_\beta(x, z)dz,$$

where $\eta_\beta(t)=\eta(t)t^{-|\beta|}(t-1)^{|\beta|}$ and $K_\beta(x,z)=D_x^{\alpha-\beta}D_z^\beta K(x,z)$. This formula and Lemma 1 in the case already proved give the estimate (3) in this case.

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