17. *On a Theorem Concerning Trigonometrical Polynomials*

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§1. H. Davenport and H. Halberstan [1] have proved the following theorem from which they have derived a generalization of theorems of K. F. Roth [2] and E. Bombieri [3] on the large sieve:

Thoerem DH1.[1] Let $S_N(x)$ be a trigonometrical polynomial of order N such that

$$S_N(x)=\sum_{n=-N}^{N} c_n e^{inx}$$

and $x_1, x_2, \cdots, x_R$ $(R\geqq 2)$ be distinct points on $(-\pi, \pi)$ such that

$$2\delta=\min_{j\neq k} |x_j-x_k|.$$

Then

$$(1)\qquad \sum_{r=1}^{R} |S_N(x_r)|^2 \leqq 4\cdot 4 \max(N, \pi/2\delta) \sum_{n=-N}^{N} |c_N|^2.$$

Our first theorem is as follows:

Theorem 1. Using the same notation as in Theorem DH1, we have

$$(2)\qquad \sum_{r=1}^{R} |S_N(x_r)|^2 \leqq A \sum_{n=-N}^{N} |c_n|^2$$

for small δ, where $A\leqq 2.34\ (N+\pi/\delta)$ or $A\leqq 3.13\ (N+\pi/2\delta)$.

The inequalities (1) and (2) are mutually exclusive. If N is near to $\pi/2\delta$, then (1) is better than (2), but if they are very different, then (2) is better than (1), except for "small δ."

Further H. Davenport and H. Halberstan [1] proved the following

Theorem DH2. Using the same notation as in Theorem DH1, we have

$$(3)\qquad \sum_{r=1}^{R} |S_N(x_r)|^p \leqq A\sqrt{p} \max(N, 2\pi/\delta)\Big(\sum_{n=-N}^{N} |c_n|^q\Big)^{p/q}$$

where A is an absolute constant and $1/p+1/q=1$, $p\geqq 2$.

Our second theorem is

Theorem 2. Using the same notation as in Theorem DH1,

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1) In [1], Theorem DH1 is stated for the trigonometrical polynomial on the interval $(0, 1)$, that is, $S_N=\sum_{n=-N}^{N} c_n e^{2\pi inx}$. Further 2δ in $(-\pi, \pi)$ corresponds to $2\delta/2\pi$ in $(0, 1)$.

(4) $$\sum_{r=1}^{R} |S_N(x_r)|^p \leqq A'(1+\varepsilon)(N+\pi/\delta)\Big(\sum_{n=-N}^{N} |c_n|^q\Big)^{p/q}$$

for any $\varepsilon>0$ and sufficiently small δ, where $1/p+1/q=1$, $p\geqq 2$ and

$$A'=\frac{2^{p-2}}{\pi^p(q+1)^{p-1}}\Big(\int_0^\infty \frac{|\sin v|^q}{v^q}dv\Big)^{p-1}\Big/\Big(\int_0^{\pi/2}\frac{\sin^2 v}{v^2}dv\Big)^p.$$

Taking $p=3, 4$, and 5, we get

(5) $$\sum_{r=1}^{R} |S_N(x_r)|^3 \leqq 0.053(1+\varepsilon)(N+\pi/\delta)\Big(\sum_{n=-N}^{N} |c_n|^{3/2}\Big)^2,$$

(6) $$\sum_{r=1}^{R} |S_N(x_r)|^4 \leqq 0.076(1+\varepsilon)(N+\pi/\delta)\Big(\sum_{n=-N}^{N} |c_n|^{4/3}\Big)^3,$$

(7) $$\sum_{r=1}^{R} |S_N(x_r)|^5 \leqq 0.143(1+\varepsilon)(N+\pi/\delta)\Big(\sum_{n=-N}^{N} |c_n|^{5/4}\Big)^4.$$

Our theorems have the application similar to [1]. For example, we have

Theorem 3. If $S_N(x)=\sum_{n=-N}^{N} c_n e^{2\pi i n x}$, then

$$\exists Q_0 : \sum_{q\leq Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} |S_N(a/q)|^2 \leqq 2.4(N+Q^2)\sum_{n=-N}^{N}|c_n|^2 \quad \text{for all} \quad Q\geqq Q_0.$$

Our method of proof of Theorem 1 and 2 is different from [1] and is adopted from our paper [4]. In § 2, we prove a formula for $S_N(x)$ which is used later. In § 3 we prove Theorem 1 and in § 4 Theorem 2 is proved.

§ 2. *General formula*. Let $f(t)$ be an integrable function having $S_N(x)$ as the Nth partial sum of its Fourier series, then

$$S_N(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)dt=\frac{1}{\pi}\int_{-\pi}^{\pi} S_N(t)D_N(x-t)dt$$

where $D_N(t)$ is the Nth Dirichlet kernel, i.e.

(8) $$D_N(t)=\frac{1}{2}+\sum_{k=1}^{N}\cos kt=\frac{\sin(N+1/2)t}{2\sin t/2}.$$

Let (λ_n) be a sequence of real numbers which are determined later, then we have the inequality (cf. [4])

$$\sum_{n=N}^{M}\lambda_n D_n(t)=\sum_{n=N}^{M}\lambda_n\Big(D_N(t)+\sum_{m=N+1}^{n}\cos nt\Big)$$
$$=\sum_{n=N}^{M}\lambda_n D_N(t)+\sum_{m=N+1}^{M}\Big(\sum_{n=m}^{M}\lambda_n\Big)\cos mt.$$

If we put $\Lambda_n=\sum_{m=1}^{n}\lambda_m$, then we get

$$\sum_{n=N}^{M}\lambda_n D_n(t)=(\Lambda_M-\Lambda_{N-1})D_N(t)+\sum_{m=N+1}^{M}(\Lambda_M-\Lambda_{m-1})\cos mt$$

and then

(9) $$D_N(t)=\frac{1}{\Lambda_M-\Lambda_{N-1}}\sum_{n=N}^{M}\lambda_n D_n(t)-\frac{1}{\Lambda_M-\Lambda_{N-1}}\sum_{n=N+1}^{M}(\Lambda_M-\Lambda_{n-1})\cos nt$$
$$=D_{N,1}(t)-D_{N,2}(t), \quad \text{say}.$$

We have, by (8),

$$D_{N,1}(t)=\frac{1}{A_M-A_{N-1}}\cdot\frac{1}{2\sin t/2}\sum_{n=N}^{M}\lambda_n\sin(n+1/2)t$$
$$=\frac{1}{A_N-A_{N-1}}\cdot\frac{1}{2\sin t/2}\mathscr{I}\Big(\sum_{n=N}^{M}\lambda_n e^{i(n+1/2)t}\Big).$$

We write $\mu=\left[\frac{1}{2}(M+N)\right]$ and $\nu=\left[\frac{1}{2}(M-N)\right]-1$ and we suppose that $\lambda_{\mu+n}=\lambda_{\mu-n}$ for $0<n\leqq\nu$ and the other λ_n vanishes, then

$$D_{N,1}(t)=\frac{1}{A_M-A_{N-1}}\,\frac{1}{2\sin t/2}\mathscr{I}\Big(e^{i(\mu+1/2)t}\sum_{n=-\nu}^{\nu}\lambda_{\mu+n}e^{int}\Big)$$

(10) $$=\frac{1}{A_M-A_{N-1}}\,\frac{\sin(\mu+1/2)t}{2\sin t/2}\Big(\lambda_\mu+2\sum_{n=1}^{\nu}\lambda_{\mu+n}\cos nt\Big)$$
$$=\frac{1}{A_M-A_{N-1}}\,D_\mu(t)\Big(\lambda_\mu+2\sum_{n=1}^{\nu}\lambda_{\mu+n}\cos nt\Big).$$

Let g be the characteristic function of the interval $(-\delta,\delta)$ with period 2π and we take (λ_n) such that $\lambda_\mu+2\sum_{n=1}^{\nu}\lambda_{\mu+n}\cos nt$ is the νth Cesàro mean of the Fourier series of g, that is,

(11) $$\lambda_\mu+2\sum_{n=1}^{\nu}\lambda_{\mu+n}\cos nt=\frac{1}{\pi}\int_{-\pi}^{\pi}g(u)K_\nu(t-u)du$$
$$=\frac{1}{\pi}\int_{-\delta}^{\delta}K_\nu(t-u)du=\frac{1}{\pi}\int_{t-\delta}^{t+\delta}K_\nu(u)du$$

where $K_\nu(u)$ is the νth Fejér kernel and is defined by

(12) $$K_\nu(u)=\frac{1}{\nu+1}\sum_{n=0}^{\nu}D_n(u)=\frac{1}{2}+\sum_{n=1}^{\nu}\Big(1-\frac{n}{\nu+1}\Big)\cos nu$$
$$=\frac{\sin^2(\nu+1)u/2}{(\nu+1)2\sin^2 u/2}$$

and then

$$\lambda_\mu+2\sum_{n=1}^{\nu}\lambda_{\mu+n}\cos nt=\frac{1}{\pi}\Big\{\delta+\sum_{n=1}^{\nu}\Big(1-\frac{n}{\nu+1}\Big)\int_{-\delta}^{\delta}\cos n(t-u)du\Big\}$$
$$=\frac{1}{\pi}\Big\{\delta+2\sum_{n=1}^{\nu}\Big(1-\frac{n}{\nu+1}\Big)\frac{\sin n\delta}{n}\cos nt\Big\}.$$

Therefore,

$$\lambda_\mu=\frac{\delta}{\pi},\qquad \lambda_{\mu+n}=\frac{1}{\pi}\Big(1-\frac{n}{\nu+1}\Big)\frac{\sin n\delta}{n}\qquad (n=1,2,\cdots,\nu)$$

and

(13) $$A_M-A_{N-1}=\lambda_\mu+2\sum_{n=1}^{\nu}\lambda_{\mu+n}=\frac{1}{\pi}\int_{-\delta}^{\delta}K_\nu(u)du=\frac{2}{\pi}\int_0^{\delta}K_\nu(u)du.$$

Now, by (10) and (11)

$$
(14)\qquad
\begin{aligned}
S_{N,1}(x)&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)D_{N,1}(x-t)dt\\
&=\frac{1}{\pi^2(\Lambda_M-\Lambda_{N-1})}\int_{-\pi}^{\pi}f(t)D_\mu(x-t)dt\int_{x-t-\delta}^{x-t+\delta}K_\nu(u)du\\
&=\frac{1}{\pi^2(\Lambda_M-\Lambda_{N-1})}\int_{-\pi}^{\pi}f(t)D_\mu(x-t)dt\int_{x-\delta}^{x+\delta}K_\nu(u-t)du\\
&=\frac{1}{\pi^2(\Lambda_M-\Lambda_{N-1})}\int_{x-\delta}^{x+\delta}du\int_{-\pi}^{\pi}f(t)D_\mu(x-t)K_\nu(u-t)dt.
\end{aligned}
$$

Further, by (9)

$$
\begin{aligned}
S_{N,2}(x)&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)D_{N,2}(x-t)dt\\
&=\frac{1}{\pi(\Lambda_M-\Lambda_{N-1})}\sum_{n=N+1}^{M}(\Lambda_M-\Lambda_{n-1})\int_{-\pi}^{\pi}f(t)\cos n(x-t)dt.
\end{aligned}
$$

If $f(t)$ is replaced by $S_N(t)$, then $S_{N,2}(t)$ vanishes and then (14) becomes

$$
(15)\qquad
\begin{aligned}
S_N(x)&=\frac{1}{\pi}\int_{-\pi}^{\pi}S_N(t)D_{N,1}(x-t)dt\\
&=\frac{1}{\pi^2(\Lambda_M-\Lambda_{N-1})}\int_{x-\delta}^{x+\delta}du\int_{-\pi}^{\pi}S_N(t)D_\mu(x-t)K_\nu(u-t)dt.
\end{aligned}
$$

We can also verify this formula directly.

§3. *Proof of Theorem* 1. We can suppose that $S_N(x)$ is real. By (15), we have

$$
\begin{aligned}
S_N^2(x)&=\frac{1}{\pi^4(\Lambda_M-\Lambda_{N-1})^2}\left\{\int_{x-\delta}^{x+\delta}du\int_{-\pi}^{\pi}S_N(t)D_\mu(x-t)K_\nu(u-t)dt\right\}^2\\
&\leqq\frac{1}{\pi^4(\Lambda_M-\Lambda_{N-1})^2}\int_{x-\delta}^{x+\delta}du\int_{-\pi}^{\pi}S_N^2(t)K_\nu^2(u-t)dt\int_{x-\delta}^{x+\delta}du\int_{-\pi}^{\pi}D_\mu^2(x-t)dt\\
&=\frac{2\delta}{\pi^4(\Lambda_M-\Lambda_{N-1})^2}\int_{-\pi}^{\pi}D_\mu^2(t)dt\int_{x-\delta}^{x+\delta}du\int_{-\pi}^{\pi}S_N^2(t)K_\nu^2(u-t)dt
\end{aligned}
$$

and then

$$
(16)\qquad
\begin{aligned}
\sum_{r=1}^{R}S_N^2(x_r)&\leqq\frac{2\delta}{\pi^4(\Lambda_M-\Lambda_{N-1})^2}\int_{-\pi}^{\pi}D_\mu^2(t)dt\int_{-\pi}^{\pi}du\int_{-\pi}^{\pi}S_N^2(t)K_\nu^2(u-t)dt\\
&=A\frac{1}{2\pi}\int_{-\pi}^{\pi}S_N^2(t)dt=A\sum_{n=-N}^{N}c_n^2,
\end{aligned}
$$

where

$$
A=\frac{4\delta}{\pi^3(\Lambda_M-\Lambda_{N-1})^2}\int_{-\pi}^{\pi}K_\nu^2(t)dt\int_{-\pi}^{\pi}D_\mu^2(t)dt.
$$

Since

$$
\frac{1}{\pi}\int_{-\pi}^{\pi}K_\nu^2(t)dt=\frac{1}{2}+\sum_{n=1}^{\nu}\left(1-\frac{n}{\nu+1}\right)^2=\frac{\nu}{3}+\frac{1}{6}+\frac{1}{3(\nu+1)}
$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_\mu^2(t)dt=\mu+\frac{1}{2}$$

by the Parseval identity and (8) and (12), we get

$$A=\frac{4\delta(\mu+1/2)}{\pi(\Lambda_M-\Lambda_{N-1})^2}\Big(\frac{\nu}{3}+\frac{1}{6}+\frac{1}{3(\nu+1)}\Big). \tag{17}$$

If we take $\nu=[\alpha/\delta]$ and suppose that δ is sufficiently small, then, by (13),

$$\Lambda_M-\Lambda_{N-1}\cong\frac{1}{\pi}\int_{-\alpha/\nu}^{\alpha/\nu}K_\nu(u)du=\frac{2}{\pi(\nu+1)}\int_0^{\alpha/\nu}\frac{\sin^2(\nu+1)u/2}{2\sin^2 u/2}du$$

$$\cong\frac{4}{\pi(\nu+1)}\int_0^{\alpha/\nu}\frac{\sin^2(\nu+1)u/2}{u^2}du\cong\frac{2}{\pi}\int_0^{\alpha/2}\frac{\sin^2 v}{v^2}dv.$$

By (17), we have

$$A\cong\frac{4(N+\alpha/\delta)}{3\pi\Big(\frac{2}{\pi}\int_0^{\alpha/2}\frac{\sin^2 v}{v^2}dv\Big)^2}\cong\frac{\alpha\pi}{3}\Big(N+\frac{\alpha}{\delta}\Big)\Big(\int_0^{\alpha/2}\frac{\sin^2 v}{v^2}dv\Big)^{-2}. \tag{18}$$

If we put $\alpha=\pi$ or $\alpha=\pi/2$ in (18), then

$$A\leqq 2.34(N+\pi/\delta)\quad\text{or}\quad A\leqq 3.13(N+\pi/2\delta),$$

respectively. This proves (2).

§4. *Proof of Theorem* 2. By (16) and the Hölder inequality, we have

$$|S_N(x)|^p\leqq\frac{1}{\pi^{2p}(\Lambda_M-\Lambda_{N-1})^p}\Big(\int_{x-\delta}^{x+\delta}du\int_{-\pi}^{\pi}|S_N(t)|^pK_\nu^p(u-t)dt\Big)\cdot\Big(\int_{x-\delta}^{x+\delta}du\int_{-\pi}^{\pi}|D_\mu(x-t)|^qdt\Big)^{p/q},$$

where $1/p+1/q=1$, and $p\geqq 2$, and then

$$\sum_{r=1}^{R}|S_N(x_r)|^p\leqq\frac{(2\delta)^{p/q}}{\pi^{2p}(\Lambda_M-\Lambda_{N-1})^p}\int_{-\pi}^{\pi}K_\nu^p(t)dt\cdot\Big(\int_{-\pi}^{\pi}|D_\mu(t)|^qdt\Big)^{p/q}\int_{-\pi}^{\pi}|S_N(t)|^pdt \tag{19}$$

By the Hausdorff-Young theorem,

$$\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}K_\nu^p(t)dt\Big)^{1/p}\leqq\frac{1}{2}\Big(1+2\sum_{n=1}^{\nu}\Big(1-\frac{n}{\nu+1}\Big)^q\Big)^{1/q}\cong\frac{1}{2^{1/p}}\frac{\nu^{1/q}}{(q+1)^{1/q}} \tag{20}$$

and

$$\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}|S_N(t)|^pdt\Big)^{1/p}\leqq\Big(\sum_{n=-N}^{N}|c_n|^q\Big)^{1/q}. \tag{21}$$

Further, we have

$$\int_{-\pi}^{\pi}|D_\mu(t)|^qdt=\frac{2}{2^q}\int_0^{\pi}\frac{|\sin(\mu+1/2)t|^q}{\sin^q t/2}dt$$

$$\leqq\frac{1}{2^{q-1}}\Big\{\Big(\frac{\eta}{\sin\eta}\Big)^q\int_0^{\eta}\frac{|\sin(\mu+1/2)t|^q}{t^q}dt+\pi^q\int_{\eta}^{\pi}\frac{|\sin(\mu+1/2)t|^q}{t^q}dt\Big\}$$

$$\leqq\frac{(\mu+1/2)^{q-1}}{2^{q-1}}\Big\{\Big(\frac{\eta}{\sin\eta}\Big)^q\int_0^{\eta(\mu+1/2)}\frac{|\sin t|^q}{t^q}dt+\pi^q\int_{\eta(\mu+1/2)}^{\pi(\mu+1/2)}\frac{|\sin t|^q}{t^q}dt\Big\}.$$

This holds for any $\eta>0$. If we take η as a fixed small number and make μ so large enough, then we get

$$\int_{-\pi}^{\pi} |D_\mu(t)|^q dt \leqq \frac{1+\varepsilon}{2^{q-1}} \mu^{q-1} \int_0^\infty \frac{|\sin t|^q}{2q-1} dt \tag{22}$$

for any fixed δ and all sufficiently large μ. Substituting (20), (21), and (22) into (19), we get

$$\begin{aligned} &\sum_{r=1}^{R} |S_N(x_r)|^p \\ &\leqq \frac{(1+\varepsilon')(2\delta\nu)^{p/q}\mu}{2\pi^{2p-1}(q+1)^{p/q}(\Lambda_M-\Lambda_{N-1})^p}\left(\int_0^\infty \frac{|\sin t|^q}{t^q} dt\right)^{p/q}\left(\sum_{n=-N}^{N} |c_n|^q\right)^{p/q} \end{aligned} \tag{23}$$

If we take $\nu=[\pi/\delta]$, then (23) becomes

$$\sum_{r=1}^{R} |S_N(x_r)|^p \leqq \frac{2^{p-2}(1+\varepsilon')(N+\pi/\delta)}{\pi^p(q+1)^{p-1}} A'' \left(\sum_{n=-N}^{N} |c_n|^q\right)^{p/q} \tag{24}$$

where

$$A''=\left(\int_0^\infty \frac{|\sin v|^q}{v^q} dv\right)^{p-1} \Big/ \left(\int_0^{\pi/2} \frac{\sin^2 v}{v^2} dv\right)^p.$$

Thus we get (4).

By the numerical calculation, we get[1)]

$$\begin{aligned} \frac{2^{p-2}A''}{\pi^p(q+1)^{p-1}} &\leqq 0.0528 \quad \text{for } p=3. \\ &\leqq 0.07576 \quad \text{for } p=4, \\ &\leqq 0.143 \quad \text{for } p=5. \end{aligned}$$

Thus we get (5), (6), and (7).

References

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