47. On the Classical Propositional Calculus of A. R. Anderson and N. D. Belnap

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In this paper, we concern with the classical propositional calculus by A. R. Anderson and N. D. Belnap [1]. In their system, axioms are formulated as "if p and $\sim p$ are in a primitive disjunction α, then α is an axiom", and rules of deduction as

$$\text{I)}\ \frac{\varphi(\alpha)}{\varphi(\sim\sim\alpha)},\qquad \text{II)}\ \frac{\varphi(\sim\alpha),\ \varphi(\sim\beta)}{\varphi(\sim(\alpha\vee\beta))}.$$

We shall show that, if we interpret $p\rightarrow q$ as $\sim p\vee q$, then we have Lukasiewicz axiom system:

1) $p\rightarrow(q\rightarrow p)$,

2) $(p\rightarrow(q\rightarrow r))\rightarrow((p\rightarrow q)\rightarrow(p\rightarrow r))$,

3) $(\sim p\rightarrow\sim q)\rightarrow(q\rightarrow p)$.

To prove 1), take

(1) $\sim p\vee(\sim q\vee p)$,

then (1) is an axiom, since (1) contains p and $\sim p$ at the same time. Hence $p\rightarrow(\sim q\vee p)$ and we have $p\rightarrow(q\rightarrow p)$.

To prove 2), it is sufficient to show

(2) $\sim(\sim p\vee\sim q\vee r)\vee\sim(\sim p\vee q)\vee\sim p\vee r$.

The following formulas are axioms:

(3) $\sim r\vee(\sim p\vee r)\vee\sim q$,

(4) $q\vee(\sim p\vee r)\vee\sim q$,

(5) $p\vee(\sim p\vee r)\vee\sim q$.

By the rule of deduction I), (4) implies

(6) $\sim\sim q\vee(\sim p\vee r)\vee\sim q$,

similarly by I), (5) implies

(7) $\sim\sim p\vee(\sim p\vee r)\vee\sim q$.

Then, by II), (3) and (6) imply

(8) $\sim(\sim q\vee r)\vee(\sim p\vee r)\vee\sim q$.

Further, by II), (7) and (8) imply

(9) $\sim(\sim p\vee(\sim q\vee r))\vee(\sim p\vee r)\vee\sim q$.

On the other hand,

(10) $\sim r\vee(\sim p\vee r)\vee p$,

(11) $q\vee(\sim p\vee r)\vee p$,

(12) $p\vee(\sim p\vee r)\vee p$.

are axioms in this system. By I), (10) implies

(13) $\sim r\vee(\sim p\vee r)\vee\sim\sim p$.

Similarly (11) implies $\sim\sim q\vee(\sim p\vee r)\vee p$ and further

(14) $\sim\sim q\vee(\sim p\vee r)\vee\sim\sim p.$

(12) implies

(15) $\sim\sim p\vee(\sim p\vee r)\vee\sim\sim p.$

By (13), (14), and II), we have

(16) $\sim(\sim q\vee r)\vee(\sim p\vee r)\vee\sim\sim p.$

We use the rule of deduction II), then we have

(17) $\sim(\sim p\vee(\sim q\vee r))\vee(\sim p\vee r)\vee\sim\sim p.$

Consequently (9) and (11) imply

$$\sim(\sim p\vee q)\vee\sim(\sim p\vee(\sim q\vee r))\vee(\sim p\vee r),$$

which is the formula (2).

Next we shall prove 3).

$$\sim p\vee(\sim q\vee p),$$
$$q\vee(\sim q\vee p)$$

are axioms in this system. Then by I), we have

$$\sim\sim\sim p\vee(\sim q\vee p)$$

and

$$\sim\sim q\vee(\sim q\vee p).$$

Then by II), we have

$$\sim(\sim\sim p\vee\sim q)\vee(\sim q\vee p),$$

which is $\sim(\sim p\rightarrow\sim q)\vee(q\rightarrow p)$. This means $(\sim p\rightarrow\sim q)\rightarrow(q\rightarrow p)$.

Therefore we complete the proof.

Reference

[1] A. R. Anderson and N. D. Belnap, Jr.: A simple treatment of truth functions. Jour. Symbolic Logic, **24**, 301-302 (1959).