46. *On the Crossed Product of Abelian von Neumann Algebras. II*

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1. This note is the continuation of [2].

Previously, we have discussed the equivalence among the groups of automorphisms of an abelian von Neumann algebra due to Dye [3] in connection with the crossed product. In the present note, we shall discuss the another notion introduced by Dye [3], *weak equivalence*, in connection with the crossed product.

We shall use the terminologies and the notations employed in [2].

2. At first, we shall introduce the definition of weak equivalence following after Dye [3].

Let $\mathscr{A}_1$ (resp. $\mathscr{A}_2$) be an abelian von Neumann algebra with the faithful normal trace ϕ_1 (resp. ϕ_2) normalized by $\phi_1(1)=1$ (resp. $\phi_2(1)=1$), and G_1 (resp. G_2) a group of ϕ-preserving automorphisms of $\mathscr{A}_1$ (resp. $\mathscr{A}_2$). Let Ψ be an isomorphism of $\mathscr{A}_1$ onto $\mathscr{A}_2$ and α_1 (resp. α_2) be an automorphism of $\mathscr{A}_1$ (resp. $\mathscr{A}_2$). Then, for $A\in\mathscr{A}_2$, $\Psi[(\Psi^{-1}(A))^{\alpha_1}]$ defines an automorphism of $\mathscr{A}_2$ which will be denoted by $\Psi(\alpha_1)$. Similarly, we can define $\Psi^{-1}(\alpha_2)$ on $\mathscr{A}_1$ by $\Psi^{-1}[\Psi(A)^{\alpha_2}]$. Under these circumstances, G_1 and G_2 are called *weakly equivalent*, if there exists an isomorphism Ψ of $\mathscr{A}_1$ onto $\mathscr{A}_2$ such that $\Psi^{-1}(G_2)=\{\Psi^{-1}(g);\ g\in G_2\}$ is equivalent to G^1 in the sense described in [2].

3. In this section, we wish to give a characterization of weak equivalence in the following

Theorem. *Let $\mathscr{A}_1$ (resp. $\mathscr{A}_2$) be an abelian von Neumann algebra, ϕ_1 (resp. ϕ_2) a normalized faithful normal trace of $\mathscr{A}_1$ (resp. $\mathscr{A}_2$), and G_1 (resp. G_2) a countable freely acting group of ϕ_1-(resp. ϕ_2-) preserving automorphisms of $\mathscr{A}_1$ (resp. $\mathscr{A}_2$). Then a necessary and sufficient condition that G_1 and G_2 are weakly equivalent is that there exists an isomorphism Φ of $G_1\otimes\mathscr{A}_1$ onto $G_2\otimes\mathscr{A}_2$ such that*

$$\Phi(\mathscr{A}_1)=\mathscr{A}_2.$$

If G_1 and G_2 are weakly equivalent, then there exists an isomorphism φ of $\mathscr{A}_1$ onto $\mathscr{A}_2$ such that $\varphi^{-1}(G_2)$ is equivalent to G_1, by the definition. Hence, by Theorem 1 in [2], there exists an isomorphism Φ_1 of $\varphi^{-1}(G_2)\otimes\mathscr{A}_1$ onto $G_1\otimes\mathscr{A}_1$ such that $\Phi_1(A)=A$ for any $A\in\mathscr{A}_1$.

Now, we shall need the following lemma which is a counterpart of [6; Remark of Lemma 1] in the case of abelian von Neumann algebras.

Lemma 1. *For* $\mathscr{A}_1, \mathscr{A}_2, G_1, G_2,$ *and* φ *as same as above, there exists an isomorphism* Φ_2 *of* $\varphi^{-1}(G_2)\otimes\mathscr{A}_1$ *onto* $G_2\otimes\mathscr{A}_2$ *such that* $\Phi_2(\mathscr{A}_1)=\mathscr{A}_2$.

We are able to trace the proof of [6; Lemma 1] in our case, we shall omit the proof of the lemma.

For these isomorphisms Φ_1 and Φ_2, the mapping $\Phi_2\circ\Phi_1^{-1}$ is evidently an isomorphism of $G_1\otimes\mathscr{A}_1$ onto $G_2\otimes\mathscr{A}_2$ which satisfies $\Phi_2\circ\Phi_1^{-1}(\mathscr{A}_1)=\mathscr{A}_2$.

Conversely, suppose that there exists an isomorphism Φ of $G_1\otimes\mathscr{A}_1$ onto $G_2\otimes\mathscr{A}_2$ such that $\Phi(\mathscr{A}_1)=\mathscr{A}_2$. For any $g\in G_2$, there is a unitary operator U_g in $G_2\otimes\mathscr{A}_2$ such that

$$U_g^*AU_g=A^g, \qquad \text{for any } A\in\mathscr{A}_2.$$

Then, $\Phi^{-1}(U_g)$ is a unitary operator in $G_1\otimes\mathscr{A}_1$ and

$$\Phi^{-1}(U_g^*\Phi(A)U_g)=\Phi^{-1}(U_g)A\Phi^{-1}(U_g), \qquad \text{for any } A\in\mathscr{A}_1.$$

Therefore $\Phi^{-1}(g)$ can be extended to an inner automorphism of $G_1\otimes\mathscr{A}_1$, whence by [1; Theorem 2] $\Phi^{-1}(g)\in[G_1]$, that is,

$$(*) \qquad [\Phi^{-1}(G_2)]\subset[[G_1]]=[G_1].$$

Likewise, we have $\Phi(G_1)\subset[G_2]$, and so

$$(**) \qquad G_1\subset\Phi^{-1}([G_2]).$$

In this place, we want to show the following

Lemma 2. *Let* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *be as same as in Theorem. If* α *(resp.* β*) is an automorphism of* $\mathscr{A}_1$ *(resp.* $\mathscr{A}_2$*) and* Ψ *is an isomorphism of* $\mathscr{A}_1$ *onto* $\mathscr{A}_2$*, then we have*

$$\Psi[F(\alpha, \Psi^{-1}(\beta)]=F(\Psi(\alpha), \beta).$$

Proof. For any projection P of $\mathscr{A}_2$ such that $P\leqq\Psi[F(\alpha,\Psi^{-1}(\beta)]$, we have

$$[\Psi^{-1}(P)]^\alpha=\Psi^{-1}(P)^{\Psi^{-1}(\beta)}=\Psi^{-1}[(\Psi(\Psi^{-1}(P)))^\beta]=\Psi^{-1}(P^\beta).$$

Hence we have

$$P^{\Psi(\alpha)}=\Psi[(\Psi^{-1}(P))^\alpha]=P^\beta,$$

and so we have finally

$$\Psi[F(\alpha, \Psi^{-1}(\beta))]\leqq F(\Psi(\alpha), \beta).$$

Similarly, we have also the converse inequality. Therefore, we have completed the proof of the lemma.

Now we shall return to the proof of the theorem. For any $\alpha\in\Phi^{-1}([G_2])$, we have $\Phi(\alpha)\in[G_2]$, and so

$$\sum_{g\in G_2}F(\Phi(\alpha), g)=1.$$

Hence, for any non-zero projection P of $\mathscr{A}_2$, there exists g of G_2 such that $F(\Phi(\alpha), g)P\neq 0$. Therefore,

$$\Phi^{-1}[F(\Phi(\alpha), g)]\Phi^{-1}(P)\neq 0.$$

Using Lemma 2, we have

$$F(\alpha, \Phi^{-1}(g))\Phi^{-1}(P) \neq 0.$$

Since P is arbitrary, we have

$$\sum_{g \in G_2} F(\alpha, \Phi^{-1}(g)) = 1,$$

that is, $\alpha \in [\Phi^{-1}(G_2)]$. Therefore, we have

$$\Phi^{-1}([G_2]) \subset [\Phi^{-1}(G_2)].$$

Similarly, we have also the converse inequality: $[\Phi^{-1}(G_2)] \subset \Phi^{-1}([G_2])$. Thus, we have

$$[\Phi^{-1}(G_2)] = \Phi^{-1}([G_2]).$$

On the other hand, we have by (**)

$$(***) \qquad [G_1] \subset [\Phi^{-1}([G_2])] = [\Phi^{-1}(G_2)].$$

The relations (*) and (***) imply

$$[G_1] = [\Phi^{-1}(G_2)].$$

Hence G_1 and G_2 are weakly equivalent, which completes the proof of Theorem.

4. In [5; Lemma 5.2.3], Murray and von Neumann stated without proof a remarkable theorem. In our language, it can be described as follows:

Let $\mathscr{A}$ be a maximal abelian von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$ without minimal non-zero projection, φ a faithful normal trace of $\mathscr{A}$ with a normalized trace vector and G a countable ergodic freely acting abelian group of φ-preserving automorphisms of $\mathscr{A}$. Then the crossed product $G \otimes \mathscr{A}$ of $\mathscr{A}$ by G is a hyperfinite II_1-factor.

They gave no indication of the proof except a remark which said "It requires some rather deep results on the decomposition of mappings of measurable sets⋯". Recently, Dye [4] succeeds in proof of the theorem of Murray and von Neumann, as a consequence of his detailed study on measure preserving transformations.

In the remainder of the note, we shall give an another proof also basing on the analysis of Dye.

In fact, by [4; Corollary 4.1] and [3; Theorem 3], there exists a freely acting ergodic group K equivalent to G, which is a set-theoretical union of a sequence $K_1 \subset K_2 \subset \cdots$ of finite groups. Then by Theorem 1 in [2] the crossed product $G \otimes \mathscr{A}$ of $\mathscr{A}$ by G is isomorphic to the crossed product $K \otimes \mathscr{A}$ of $\mathscr{A}$ by K.

On the other hand, by the well-known theorem [5; Lemma 5.2.2], the crossed product $K \otimes \mathscr{A}$ is a hyperfinite factor. Therefore, the crossed product $G \otimes \mathscr{A}$ is a hyperfinite II_1-factor.

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