45. Axiom Systems of Aristotle Traditional Logic. II

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In this paper, we shall give new axiom systems of Aristotle traditional logic. Some systems have been obtained by J. Lukasiewicz ([4], [5]), I. Bocheński ([1]), N. Kretzmann ([3]), and recently K. Iséki ([2]).

K. Iséki has given a method to find axiom systems. For the detail, see [2].

We use the following notations. For the categorical sentences,

1) Aab : Every a is b,
2) Iab : At least one a is b,
3) Oab : At least one a is not b,
4) Eab : No a is b,

For functors,

1) C : Implication, 2) N : Nagation, 3) K : Conjunction.

Then we have

$D1$ $Eab=NIab$, $D2$ $Oab=NAab$.

For moods and figures:

1) XY_1 : $CXabYab$,
2) XY_2 : $CXabYba$,
3) XYZ_1 : $CKXabYcaZcb$,
4) XYZ_2 : $CKXabYcbZca$,
5) XYZ_3 : $CKXabYacZcb$,
6) XYZ_4 : $CKXabYbcZca$.

Under these symbols, the Lukasiewicz axiom system is written in the form of

$L1$ Aaa,
$L2$ Iaa,
$L3$ AAA_1,
$L4$ AII_3.

From theses of the classical propositional calculus, we have the following deduction rules $T1$–$T7$. We shall symbolize these rules as right sides.

$T1$ $CK\alpha\beta\gamma \longrightarrow CK\beta\alpha\gamma$: $\alpha\beta\gamma \longrightarrow$ (i) $\beta\alpha\gamma$,
$\searrow CK\alpha N\gamma N\beta$: $\searrow$ (ii) $\alpha N\gamma N\beta$,
$\searrow CKN\gamma\beta N\alpha$: $\searrow$ (iii) $N\gamma\beta N\alpha$,

$T2$ $CK\alpha\beta\gamma,\ C\delta\alpha \longrightarrow CK\delta\beta\gamma$; $\alpha\beta\gamma+\delta\alpha \longrightarrow \delta\beta\gamma$,
$T3$ $CK\alpha\beta\gamma,\ C\delta\beta \longrightarrow CK\alpha\delta\gamma$; $\alpha\beta\gamma+\delta\beta \longrightarrow \alpha\delta\gamma$,
$T4$ $CK\alpha\beta\gamma,\ C\gamma\delta \longrightarrow CK\alpha\beta\delta$; $\alpha\beta\gamma+\gamma\delta \longrightarrow \alpha\beta\delta$,

$T5$ $\quad C\alpha\beta \longrightarrow CN\beta N\alpha;$ $\qquad \alpha\beta \longrightarrow N\beta N\alpha,$

$T6$ $\quad CK\alpha\beta\gamma,\ \alpha \longrightarrow C\beta\gamma;$ $\qquad \alpha\beta\gamma + \alpha \longrightarrow \beta\gamma,$

$T7$ $\quad CK\alpha\beta\gamma,\ \beta \longrightarrow C\alpha\gamma;$ $\qquad \alpha\beta\gamma + \beta \longrightarrow \alpha\gamma.$

Theorem 1. *Under the deduction rule $T1$, we have the following deductively equivalent groups $G1$–$G8$:*

$G1$ $\quad AAA_1 \quad AOO_2 \quad OAO_3,$

$G2$ $\quad EAE_1 \quad EIO_2 \quad IAI_3,$

$G3$ $\quad AAI_1 \quad AEO_2 \quad EAO_3,$

$G4$ $\quad EAO_1 \quad EAO_2 \quad AAI_3,$

$G5$ $\quad EIO_1 \quad EAE_2 \quad AII_3,$

$G6$ $\quad AII_1 \quad AEE_2 \quad EIO_3,$

$G7$ $\quad AEE_4 \quad EIO_4 \quad IAI_4,$

$G8$ $\quad AAI_4 \quad AEO_4 \quad EAO_4.$

Proof. Under the deduction rule $T1$, we have

$R1$ $\quad XYZ_1 \longrightarrow$ (i) $XNZNY_2$ ($T1$ (ii)),
$\qquad\qquad \searrow$ (ii) $NZYNX_3$ ($T1$ (iii)),

$R2$ $\quad XYZ_2 \longrightarrow$ (i) $XNZNY_1$ ($T1$ (ii)),
$\qquad\qquad \searrow$ (ii) $YNZNX_3$ ($T1$ (iii), $T1$ (i)),

$R3$ $\quad XYZ_3 \longrightarrow$ (i) $NZXNY_2$ ($T1$ (ii), $T1$ (i)),
$\qquad\qquad \searrow$ (ii) $NZYNX_1$ ($T1$ (iii)),

$R4$ $\quad XYZ_4 \longrightarrow$ (i) $NZXNY_4$ ($T1$ (ii), $T1$ (i)),
$\qquad\qquad \searrow$ (ii) $YNZNX_4$ ($T1$ (iii), $T1$ (i)).

For example, we can change AAA_1 into the following thesis containing N by $R1$, and further can eliminate the functor N, by $D1$.

$ANANA_2 \longrightarrow AOO_2$ ($R1$ (i), $D1$),

$NAANA_3 \longrightarrow OAO_3$ ($R1$ (ii), $D1$).

Similarly we have

$AOO_2 \longrightarrow ANONO_1 \longrightarrow AAA_1$ ($R2$ (i), $D1$),
$\qquad \searrow NOONA_3 \longrightarrow OAO_3$ ($R2$ (ii), $D1$),

$OAO_3 \longrightarrow ONONA_2 \longrightarrow AOO_2$ ($R3$ (i), $D1$),
$\qquad \searrow NOANO_1 \longrightarrow AAA_1$ ($R3$ (ii), $D1$).

Therefore AAA_1, AOO_2, and OAO_3 are deductively equivalent each other under $T1$. By the same method, we have

$EAE_1 \longrightarrow ENENA_2 \longrightarrow EIO_2$ ($R1$ (i), $D1$, $D2$),
$\qquad \searrow NEANE_3 \longrightarrow IAI_3$ ($R1$ (ii), $D1$),

$EIO_2 \longrightarrow ENONI_1 \longrightarrow EAE_1$ ($R2$ (i), $D1$, $D2$),
$\qquad \searrow INONE_3 \longrightarrow IAI_3$ ($R2$ (ii), $D1$, $D2$),

$IAI_3 \longrightarrow NIINA_2 \longrightarrow EIO_2$ ($R3$ (i), $D1$, $D2$),
$\qquad \searrow NIANI_1 \longrightarrow EAE_1$ ($R3$ (ii), $D1$),

$AAI_1 \longrightarrow ANINA_2 \longrightarrow AEO_2$ ($R1$ (i), $D1$, $D2$),
$\qquad \searrow NIANA_3 \longrightarrow EAO_3$ ($R1$ (ii), $D1$, $D2$),

$AEO_2 \longrightarrow ANONE_1 \longrightarrow AAI_1$ ($R2$ (i), $D1$, $D2$),
$\qquad \searrow ENONA_3 \longrightarrow EAO_3$ ($R2$ (ii), $D2$),

$EAO_3 \longrightarrow NOENA_2 \longrightarrow AEO_2$ ($R3$ (i), $D2$),
$\qquad \searrow NOANE_1 \longrightarrow AAI_1$ ($R3$ (ii), $D1$, $D2$),

$EAO_1 \longrightarrow ENONA_2 \longrightarrow EAO_2$ ($R1$ (i), $D2$),
$\searrow NOANE_3 \longrightarrow AAI_3$ ($R1$ (ii), $D1$, $D2$),
$EAO_2 \longrightarrow ENONA_1 \longrightarrow EAO_1$ ($R2$ (i), $D2$),
$\searrow ANONE_3 \longrightarrow AAI_3$ ($R2$ (ii), $D1$, $D2$),
$AAI_3 \longrightarrow NIANA_2 \longrightarrow EAO_2$ ($R3$ (i), $D1$, $D2$),
$\searrow NIANA_1 \longrightarrow EAO_1$ ($R3$ (ii), $D1$, $D2$),
$EIO_1 \longrightarrow ENONI_2 \longrightarrow EAE_2$ ($R1$ (i), $D1$, $D2$),
$\searrow NOINE_3 \longrightarrow AII_3$ ($R1$ (ii), $D1$, $D2$),
$EAE_2 \longrightarrow ENENA_1 \longrightarrow EIO_1$ ($R2$ (i), $D1$, $D2$),
$\searrow ANENE_3 \longrightarrow AII_3$ ($R2$ (ii), $D1$),
$AII_3 \longrightarrow NIANI_2 \longrightarrow EAE_2$ ($R3$ (i), $D1$),
$\searrow NIINA_1 \longrightarrow EIO_1$ ($R3$ (ii), $D1$, $D2$),
$AII_1 \longrightarrow ANINI_2 \longrightarrow AEE_2$ ($R1$ (i), $D1$),
$\searrow NIINA_3 \longrightarrow EIO_3$ ($R1$ (ii), $D1$, $D2$),
$AEE_2 \longrightarrow ANENE_1 \longrightarrow AII_1$ ($R2$ (i), $D1$),
$\searrow ENENA_3 \longrightarrow EIO_3$ ($R2$ (ii), $D1$, $D2$),
$EIO_3 \longrightarrow NOENI_2 \longrightarrow AEE_2$ ($R3$ (i), $D1$, $D2$),
$\searrow NOINE_1 \longrightarrow AII_1$ ($R3$ (ii), $D1$, $D2$),
$AEE_4 \longrightarrow NEANE_4 \longrightarrow IAI_4$ ($R4$ (i), $D1$),
$\searrow ENENA_4 \longrightarrow EIO_4$ ($R4$ (ii), $D1$, $D2$),
$EIO_4 \longrightarrow NOENI_4 \longrightarrow AEE_4$ ($R4$ (i), $D1$, $D2$),
$\searrow INONE_4 \longrightarrow IAI_4$ ($R4$ (ii), $D1$, $D2$),
$IAI_4 \longrightarrow NIINA_4 \longrightarrow EIO_4$ ($R4$ (i), $D1$, $D2$),
$\searrow ANINI_4 \longrightarrow AEE_4$ ($R4$ (ii), $D1$),
$AAI_4 \longrightarrow NIANA_4 \longrightarrow EAO_4$ ($R4$ (i), $D1$, $D2$),
$\searrow ANINA_4 \longrightarrow AEO_4$ ($R4$ (ii), $D1$, $D2$),
$AEO_4 \longrightarrow NOANE_4 \longrightarrow AAI_4$ ($R4$ (i), $D1$, $D2$),
$\searrow ENONA_4 \longrightarrow EAO_4$ ($R4$ (ii), $D2$),
$EAO_4 \longrightarrow NOENA_4 \longrightarrow AEO_4$ ($R4$ (i), $D2$),
$\searrow ANONE_4 \longrightarrow AAI_4$ ($R4$ (ii), $D1$, $D2$).

Therefore the proof is complete.

Theorem 2. *The Aristotle traditional logic is given by the following set:*

$L1$, $L2$, *a thesis of* $G1$, *and a thesis of* $G5$.

Proof. An axiom system of J. Lukasiewicz [5] is given by $L1$, $L2$, $L3$, and $L4$. $L3$ is a thesis of $G1$, and $L4$ is a thesis of $G5$. Therefore, by Theorem 1, we have Theorem 2.

Theorem 3. *The Aristotle traditional logic is given by the following set:*

$L1$, $L2$, *a thesis of* $G1$, *and a thesis of* $G7$.

Proof.

$L1$ Aaa,

$L2$ Iaa,

$G1$ $AAA_1 \sim AOO_2 \sim OAO_3$,

$G7$ $IAI_4 \sim EIO_4 \sim AEE_4$.

From the rule of term order changes (see, [2]), we have

$RT1$ $\quad XYZ_1 \longrightarrow$ (i) $\tilde{X}YZ_2$, $\searrow$ (ii) $X\tilde{Y}Z_3$,

$RT2$ $\quad XYZ_2 \longrightarrow$ (i) $\tilde{X}YZ_1$, $\searrow$ (ii) $X\tilde{Y}Z_4$,

$RT3$ $\quad XYZ_3 \longrightarrow$ (i) $\tilde{X}YZ_4$, $\searrow$ (ii) $X\tilde{Y}Z_1$,

$RT4$ $\quad XYZ_4 \longrightarrow$ (i) $\tilde{X}YZ_1$, $\searrow$ (ii) $X\tilde{Y}Z_3$,

where $\tilde{X}$ means the order change of terms contained in X.

In $T6$, put $\alpha = Iaa$, $\beta = Aab$, $\gamma = Iba$, then by $G7$ and $L2$, we have $AabIba$. That is:

$IaaAI_4 + Iaa \longrightarrow 1$, ($T6$, $G7$, $L2$)

1 $\quad AI_2$.

$IAaaI_4 + Aaa \longrightarrow 2$, ($T7$, $G7$, $L1$)

2 $\quad II_2$.

$AI_2 \longrightarrow 3$, ($T5$, 1, $D1$, $D2$)

3 $\quad EO_2$.

$II_2 \longrightarrow 4$, ($T5$, 2, $D1$)

4 $\quad EE_2$.

$EIO_4 + AI_2 \longrightarrow 5$, ($T3$, $G7$, 1, $RT4$ (ii))

5 $\quad EAO_2 \sim AAI_3$.

$AAaaI_3 + Aaa \longrightarrow 6$, ($T7$, 5, $L1$)

6 $\quad AI_1$.

$AI_1 \longrightarrow 7$, ($T5$, 6, $D1$, $D2$)

7 $\quad EO_1$.

$IAI_4 + II_2 \longrightarrow 8$, ($T2$, $G7$, 2, $RT4$ (i))

8 $\quad IAI_3$.

$IAbbI_3 + Abb \longrightarrow 9$, ($T7$, 8, $L1$)

9 $\quad II_1$.

$II_1 \longrightarrow 10$, ($T5$, 9, $D1$)

10 $\quad EE_1$.

$EIO_4 + EE_2 \longrightarrow 11$, ($T2$, $G7$, 4, $RT4$ (i))

11 $\quad EIO_3$.

$EIO_3 + II_2 \longrightarrow 12$, ($T3$, 11, 2, $RT3$ (ii))

12 $\quad EIO_1 \sim AII_3 \sim EAE_2$.

$L1$, $L2$, $L3$ (one of $G1$), and 12 are axioms by J. Lukasiewicz, I. Bocheński and K. Iséki. Therefore the proof is complete.