105. On Certain Condition for the Principle of Limiting Amplitude. II

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1. Introduction and results. We consider the problem

$$\left[\frac{\partial^2}{\partial t^2} - \Delta + q(x)\right]u(x,t)=0 \qquad (t>0), \tag{1}$$

$$u(x,0)=0, \qquad \frac{\partial u}{\partial t}(x,0)=f(x),$$

where x is a point of 3-dimensional Euclidean space $E=R^3$, and Δ denotes the Laplace operator in E.

In an earlier paper [1], for the case that q has compact support we proved that under the certain condition the principle of limit amplitude for the problem (1) is valid if and only if there exists no solution $\omega \notin L^2(E)$ of the equation $(-\Delta+q)\omega=0$ satisfying conditions $\omega=O(|x|^{-1})$, $\dfrac{\partial \omega}{\partial x_i}=O(|x|^{-2})$ $(|x|\to\infty)$ (see [2]).

In the present paper we shall prove the same one for the case that the support of q is not compact.

Through the present paper $q(x)$ and $f(x)$ are assumed to satisfy the following conditions (C_1), (C_2), and (C_3):

(C_1) *$q(x)$ is a locally Hölder continuous real-valued function and behaves like $O(|x|^{-2-\alpha})$ $(\alpha>0)$ at infinity.*

By A we denote the unique self-adjoint extension in $L^2(E)$ of $-\Delta+q$ defined on $C_0^\infty(E)$.

(C_2) *A has no eigenvalue.*

Then A is positive definite.

(C_3) *f belongs to the domain $D(A^{\frac{1}{2}})$ of the self-adjoint operator $A^{\frac{1}{2}}$ and behaves like $O(|x|^{-3-\alpha})$ at infinity.*

Under the assumptions (C_1), (C_2), and (C_3) we have the followings:

Theorem 1. *Suppose that $\langle f, \omega\rangle=0$, where ω is the preceding one and $\langle f, \omega\rangle$ denotes $\int_E f(x)\omega(x)dx$. Then for the solution $u(t)\equiv u(x,t)$ of (1) we have*

$$\lim_{t\to\infty}(u(t),\varphi)_{L^2(E)}=0 \qquad \text{for all } \varphi\in L^2(E),$$

and

$$\lim_{t\to\infty}\|u(t)\|_{L^2(K)}=0 \qquad \text{for all compact } K\subset E.$$

Theorem 2. *Suppose that* $q \in C^2(E)$ *and* $q=O(|x|^{-3-\alpha})$, $D^\beta q = O(|x|^{-2-\alpha})$ $(|x|\to\infty)$ $(|\beta|=1,2)$. *Then the solution of* (1) *is such that for any* $\varphi \in L^2(E)$ *satisfying the condition* $\varphi=O(|x|^{-3-\alpha})$ $(|x|\to\infty)$ *we have*

$$\lim_{t\to\infty}\langle u(t),\varphi\rangle = 4\pi\langle\varphi,\omega\rangle\langle f,\omega\rangle\langle q,w\rangle^{-1},$$

where ω *is the above one.*

2. **Proof of Theorem 1.** Let us define an operator for functions in $L^6(E)$ by $T\varphi(x) = -\frac{1}{4\pi}\int_E \frac{q(y)\varphi(y)}{|x-y|}dy$ $(\varphi \in L^6)$. Then by virtue of Lemma 3.2 in [4] we have

Lemma 1. 1) *T is a compact operator on L^6 and the adjoint operator T^* of T with respect to the inner product $\langle\ ,\ \rangle$ is a compact operator on $L^{\frac{6}{5}}$ given as follows:*

$$T^*\omega'(x) = -\frac{1}{4\pi}q(x)\int_E \frac{\omega'(y)}{|x-y|}dy \qquad (\omega' \in L^{\frac{6}{5}}).$$

2) *By M, M' we denote the subspaces $\{\omega \in L^6; (I-T)\omega=0\}$, $\{\omega' \in L^{\frac{6}{5}}; (I-T^*)\omega'=0\}$ of $L^6, L^{\frac{6}{5}}$ respectively. Then we have that $\dim M = \dim M' \leqq 1$ and that $\langle q,\omega\rangle \neq 0$ for $\omega \in M$ $(\omega\neq 0)$. Furthermore, for $\omega \in M$ we have that $\omega \in C^2(E)$, $\omega=O(|x|^{-1})$, $\frac{\partial\omega}{\partial x_i}=O(|x|^{-2})$ $(|x|\to\infty)$ and for $\omega' \in M'$ we have that $\omega' \in C^0(E)$, $\omega'=O(|x|^{-3-\alpha})$ $(|x|\to\infty)$.*

By virtue of Lemma 1 and Riesz-Schauder's theory we have

Lemma 2. *Suppose that $\varphi \in L^2(E)$, $\varphi=O(|x|^{-3-d})$ $(|x|\to\infty)$, and $\langle\varphi,\omega\rangle=0$ for $\omega \in M$. Then we have that $\varphi \in R(A^{\frac{1}{2}})$, where $R(A^{\frac{1}{2}})$ denotes therange of $A^{\frac{1}{2}}$.*

Proof of Theorem 1. It follows from Lemma 2 and theorem 6 in [4] that $\lim_{t\to\infty}(u(t),\varphi)_{L^2(E)}=0$ for all $\varphi \in L^2(E)$.

Lemma 2 and the first part of Theorem 1 and an argument similar to the one used in proving Lemma 4.1 in [5] give that $\lim_{t\to\infty}\|u(t)\|_{L^2(K)}=0$ for all compact $K\subset E$.

3. **Proof of Theorem 2.** Suppose that there exist functions $\omega \in M$ such that $\omega\neq 0$. Then 2) of Lemma 1 implies that $\dim M=1$. Therefore, taking $\omega \in M$ such that $\langle q,\omega\rangle=1$, we have only to prove

$$(2) \qquad \lim_{t\to\infty}\langle u(t),q\rangle = 4\pi\langle f,\omega\rangle.$$

To this we use the following

Lemma 3. *Let $a>0$. Then $u(x,t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{\zeta t}R(-\zeta^2)fd\zeta$ is the solution of the problem* (1), *where $R(-\zeta^2)f$ denotes $(A+\zeta^2)^{-1}f$.*

Now we shall prove (2). Since A has no eigenvalue, by virtue of theorem 6 in [4] we see that $\frac{d}{d\lambda}\langle E_\lambda f, q\rangle \in L^1(0, \infty)$, where E_λ is the resolution of the identity generated by the operator A. Therefore by virtue of Lemma 3 and Fubini's theorem we have

$$\langle u(t), q\rangle = \frac{1}{2\pi i}\int_0^\infty \frac{d}{d\lambda}\langle E_\lambda f, q\rangle d\lambda \int_{a-i\infty}^{a+i\infty} \frac{e^{\zeta t}}{\lambda+\zeta^2} d\zeta$$

$$= \frac{1}{2\pi i}\int_0^\infty \frac{d}{d\lambda}\langle E_\lambda f, q\rangle d\lambda \int_{\Gamma_1+\Gamma_2} \frac{e^{\zeta t}}{\lambda+\zeta^2} d\zeta + \int_{N^2}^\infty \frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}} d\langle E_\lambda f, q\rangle$$

for $N > 2a$,

where Γ_1 and Γ_2 are the curves

$\{s-iN;\ 0<s\leqq a\} \cup \{a+is;\ -N<s<N\} \cup \{s+iN;\ 0<s\leqq a\}$,

$\{s+iN;\ -a\leqq s<0\} \cup \{-a+is;\ -N<s<N\} \cup \{s-iN;\ -a\leqq s<0\}$

taken in the positive direction.

We can take N so large that $\left|\int_{N^2}^\infty \frac{\sin\sqrt{\lambda}\, t}{\sqrt{\lambda}} d\langle E_\lambda f, q\rangle\right|$ becomes sufficiently small uniformly with respect to $t>0$. Let N fix sufficiently large. Since on Γ_2, $\operatorname{Re}\zeta<0$, by virtue of Lebesque's theorem we have

$$\lim_{t\to\infty}\int_0^\infty \frac{d}{d\lambda}\langle E_\lambda f, q\rangle d\lambda \int_{\Gamma_2} \frac{e^{\zeta t}}{\lambda+\zeta^2} d\zeta = 0.$$

Consequently we have only to prove

$$(3)\qquad \lim_{t\to\infty}\int_0^\infty \frac{d}{d\lambda}\langle E_\lambda f, q\rangle d\lambda \int_{\Gamma_1} \frac{e^{\zeta t}}{\lambda+\zeta^2} d\zeta = 8\pi^2 i\langle f, \omega\rangle.$$

Since we have that $R(-\zeta^2)f = \psi_\zeta + T_\zeta R(-\zeta^2)f$ and $\langle R(-\zeta^2)f, q\rangle = 4\pi\frac{1}{\zeta}\langle -\psi_\zeta, q\omega\rangle + \zeta\langle R(-\zeta^2)f, p(\zeta)\rangle$, by virtue of Fubini's theorem we have

$$(4)\qquad \int_0^\infty \frac{d}{d\lambda}\langle E_\lambda f, q\rangle d\lambda \int_{\Gamma_1} \frac{e^{\zeta t}}{\lambda+\zeta^2} d\zeta$$

$$= 4\pi\langle f, \omega\rangle \int_{\Gamma_1} \frac{e^{\zeta t}}{\zeta} d\zeta + \int_{\Gamma_1} e^{\zeta t} F(\zeta) d\zeta + \int_{\Gamma_1} \zeta e^{\zeta t}\langle R(-\zeta^2)f, T_\zeta^{*3} p(\zeta)\rangle d\zeta.$$

Here

$$F(\zeta) = \int f(y) q(x)\omega(x) dx dy \int_0^1 e^{-\zeta|x-y|\tau} d\tau + \zeta \sum_{j=0}^{2} \langle T_\zeta^j \psi_\zeta, p(\zeta)\rangle,$$

$$p(x, \zeta) = q(x)\int q(y)\omega(y)\, |x-y|\, dx dy \int_0^1 d\tau' \int_0^1 \tau e^{-\zeta|x-y|\tau\tau'} d\tau,$$

$$\psi_\zeta(x) = \frac{1}{4\pi}\int \frac{e^{-\zeta|x-y|}}{|x-y|} f(y) dy,$$

$$T_\zeta \psi(x) = -\frac{1}{4\pi}\int \frac{e^{-\zeta|x-y|}}{|x-y|} q(y)\psi(y) ay,$$

$$T_\zeta^* \psi(x) = -\frac{1}{4\pi} q(x) \int \frac{e^{-\zeta|x-y|}}{|x-y|} \psi(y) dy,$$

$$T^0 \psi(x) = \psi(x), \qquad T^j \psi(x) = T(T^{j-1}\psi)(x) \qquad (j=1, 2, 3).$$

Then without difficulty we have

$$(5) \qquad \lim_{t\to\infty} 4\pi \langle f, \omega\rangle \int_{\Gamma_1} \frac{e^{\zeta t}}{\zeta} d\zeta = 8\pi^2 i \langle f, \omega\rangle,$$

$$(6) \qquad \lim_{t\to\infty} \int_{\Gamma_1} \frac{e^{\zeta t}}{\zeta} F(\zeta) d\zeta = 0.$$

Therefore we have only to prove the following

$$(7) \qquad \lim_{t\to\infty} \int_{\Gamma_1} \zeta e^{\zeta t} \langle R(-\zeta^2) f, p_3(\zeta)\rangle d\zeta = 0,$$

where $p_3(x, \zeta) = T_\zeta^{*3} p(x, \zeta)$.

To do it we use the following

Lemma 4. *For $\lambda > 0$ we set $\theta(\lambda) \equiv \theta(x, \lambda) = \frac{1}{2\pi i}(u_+(x, \lambda) - u_-(x, \lambda))$, where $u_\pm(x, \lambda) = R(\lambda \pm i0) f(x)$. By $C_{3+\alpha}^2$ we denote the Banach space $\{\varphi \in C^2(E), \sup_{x\in E, |\beta|\leq 2} |D^\beta \varphi(x)| (1+|x|^2)^{\frac{3+\alpha}{2}} < \infty\}$ with the norm $\|\varphi\|_{C_{3+\alpha}^2} = \sup_{x\in E, |\beta|\leq 2} |D^\beta \varphi(x)| (1+|x|^2)^{\frac{3+\alpha}{2}}$. Then $T_\lambda(\varphi) \equiv \langle \theta(\lambda), \varphi\rangle$ ($\varphi \in C_{3+\alpha}^2$) is a nuclear operator from $C_{3+\alpha}^2$ to $L_\lambda^1(0, \infty)$ and $\|T_\lambda\|_{(C_{3+\alpha}^2)^*} = \|\theta(\lambda)\|_{(C_{3+\alpha}^2)^*}$ belongs to $L_\lambda^1(0, \infty)$.*

Proof of (7). By virtue of Lemma 4 and Fubini's theorem we have

$$(8) \qquad \int_{\Gamma_1} \zeta e^{\zeta t} \langle R(-\zeta^2) f, p_3(\zeta)\rangle d\zeta = \lim_{\varepsilon\to 0} \int_0^\infty d\lambda \int_{\Gamma_\varepsilon} \frac{\langle \theta(\lambda), p_3(\zeta)\rangle}{\lambda+\zeta^2} \zeta e^{\zeta t} d\zeta,$$

where Γ_ε is the path obtained replacing a by ε in Γ_1. Furthermore by virtue of Lemma 4, Lebesque's theorem, theorem 4 in [3] and Riemann-Lebesque's theorem we see that we have only to prove

$$(9) \qquad \lim_{t\to\infty} \lim_{\varepsilon\to 0} \int_0^{4N^2} d\lambda \int_{\varepsilon-iN}^{\varepsilon+iN} \frac{\langle \theta(\lambda), p_3(\zeta)\rangle}{\lambda+\zeta^2} \zeta e^{\zeta t} d\zeta = 0.$$

To this we have only to prove

$$(10) \qquad \lim_{t\to\infty} \lim_{\varepsilon\to 0} \int_0^{4N^2} d\lambda \int_{-N}^{N} e^{(\varepsilon+is)t} \frac{(\lambda-s)\langle \lambda\theta(\lambda^2), p_3(\varepsilon+is)\rangle}{(\lambda-s)^2+\varepsilon^2} ds = 0.$$

Set $\rho = t - (|x-y| + |y-z| + |z-u| + |u-v|\tau\tau')$. Then by virtue of Fubini's theorem, for fixed $t>0$ and fixed $\varepsilon>0$ we have

$$(11) \qquad \int_{-N}^{N} e^{(\varepsilon+is)t} \frac{(\lambda-s)\langle \lambda\theta(\lambda^2), p_3(\varepsilon+is)\rangle}{(\lambda-s)^2+\varepsilon^2} ds = \left(\frac{1}{4\pi}\right)^3 e^{\varepsilon t} \int \lambda\theta(x, \lambda^2) \varphi_{\varepsilon,t}(x) dx,$$

where

$$(12) \qquad \varphi_{\varepsilon,t}(x) = q(x) \int \frac{q(y)}{|x-y|} dy \int \frac{q(z)}{|y-z|} dz \int \frac{q(u)}{|z-u|} du$$
$$\times \int |u-v| q(v)\omega(v) dv \int_0^1 d\tau' \int_0^1 \tau e^{-\varepsilon(t-\rho)} d\tau \int_{-N}^{N} \frac{(s-\lambda) e^{i\rho s}}{(s-\lambda)^2+\varepsilon^2} ds.$$

First we shall prove

$$\lim_{\varepsilon\to 0}\int_0^N d\lambda\int_{-N}^N e^{(\varepsilon+is)t}\frac{(\lambda-s)\langle\lambda\theta(\lambda^2),\, p_3(\varepsilon+is)\rangle}{(\lambda-s)^2+\varepsilon^2}ds = \left(\frac{1}{4\pi}\right)^3\int_0^N d\lambda\int\lambda\theta(x,\lambda^2)\lim_{\varepsilon\to 0}\varphi_{\varepsilon,t}(x)dx. \tag{13}$$

Let $t>0$ be fixed. Then we see that there exists a constant C such that for any $\lambda<N$ we have

$$\sup_{x\in E,|\beta|\leqq 2}|D^\beta\varphi_{\varepsilon,t}(x)|(1+|x|^2)^{\frac{3+\alpha}{2}}\leqq C\left(1+\log\frac{N+\lambda}{N-\lambda}\right) \tag{14}$$

for all $\varepsilon\leqq\varepsilon_0$.

In fact, since $s\cos s$ is an odd function, for $\lambda<N$ we have

$$\int_{-N}^N\frac{(s-\lambda)e^{i\rho s}}{(s-\lambda)^2+\varepsilon^2}ds = e^{i\lambda\rho}\left[\int_{(-N-\lambda)\rho}^{(\lambda-N)\rho}\frac{\cos s}{s}ds-\varepsilon^2\rho^2\int_{(-N-\lambda)\rho}^{(\lambda-N)\rho}\frac{\cos s}{s(s^2+\varepsilon^2\rho^2)}ds + i\int_{(-N-\lambda)\rho}^{(N-\lambda)\rho}\frac{\sin s}{s}ds-i\varepsilon^2\rho^2\int_{(-N-\lambda)\rho}^{(N-\lambda)\rho}\frac{\sin s}{s(s^2+\varepsilon^2\rho^2)}ds\right]. \tag{15}$$

Therefore by virtue of the second mean value theorem for the Riemann integral we have

$$\left|\int_{-N}^N\frac{(s-\lambda)e^{i\rho s}}{(s-\lambda)^2+\varepsilon^2}ds\right|\leqq C'\left(1+\log\frac{N+\lambda}{N-\lambda}\right), \tag{16}$$

where C' is a constant independent of ε. Since $q=O(|x|^{-3-\alpha})$, $D^\beta q=O(|x|^{-2-\alpha})$ $(|x|\to\infty)$ $(|\beta|=1, 2)$, and $t-\rho\geqq 0$, by means of (12) and (16) we get (14). By virtue of (11), (14), Lemma 4, theorem 5 in [3] and Lebesque's theorem we get (13).

By virtue of (12), (15), (16) and Lebesque's theorem for $\lambda<N$ we have

$$\begin{aligned}\varphi_t(x)\equiv\lim_{\varepsilon\to 0}\varphi_{\varepsilon,t}(x) &= q(x)\int\frac{q(y)}{|x-y|}dy\int\frac{q(z)}{|y-z|}dz\int\frac{q(u)}{|z-u|}du\\ &\times\int|u-v|\,q(v)\omega(v)dv\int_0^1 d\tau'\int_0^1\tau e^{i\lambda\rho}d\tau\\ &\times\left[\int_{(-N-\lambda)\rho}^{(\lambda-N)\rho}\frac{\cos s}{s}ds+i\pi+i\left(\int_{(-N-\lambda)\rho}^{(N-\lambda)\rho}\frac{\sin s}{s}ds-\pi\right)\right]\\ &\equiv J_1+J_2+J_3.\end{aligned} \tag{17}$$

Now we shall prove

$$\lim_{t\to\infty}\int_0^N d\lambda\int\lambda\theta(x,\lambda^2)\varphi_t(x)dx=0. \tag{18}$$

Since $\rho=t-(|x-y|+|y-z|+|z-u|+|u-v|\tau\tau')$, by virtue of Lemma 4 and Riemann-Lebesque's theorem we have

$$\lim_{t\to\infty}\int_0^N d\lambda\int\lambda\theta(x,\lambda^2)J_2dx=0. \tag{19}$$

Let $\rho-t$ be fixed. Then we see that $\rho\to\infty$ as $t\to\infty$. Consequently an argument similar to the one used in proving (13) gives (20):

$$\lim_{t\to\infty}\int_0^N d\lambda \int \lambda\theta(x, \lambda^2) J_k dx = 0 \qquad (k=1, 3). \tag{20}$$

By means of (17), (19), and (20) we get (18), which gives

$$\lim_{t\to\infty}\lim_{\varepsilon\to 0}\int_0^N d\lambda \int_{-N}^N e^{(\varepsilon+is)t}\frac{(\lambda-s)\langle\lambda\theta(\lambda^2), p_3(\varepsilon+is)\rangle}{(\lambda-s)^2+\varepsilon^2}ds = 0. \tag{21}$$

In the same way we have

$$\lim_{t\to\infty}\lim_{\varepsilon\to 0}\int_N^{2N} d\lambda \int_{-N}^N e^{(\varepsilon+is)t}\frac{(\lambda-s)\langle\lambda\theta(\lambda^2), p_3(\varepsilon+is)\rangle}{(\lambda-s)^2+\varepsilon^2}ds = 0.$$

This and (21) gives (10). Thus the desired equality (7) is proved.

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