102. *On Diffeomorphisms of the n-Disk*[*)]

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1. Introduction. Let D^n denote the closed unit n-disk in R^n and let $\mathrm{Diff}(D^n)$ denote the group of orientation preserving C^1 diffeomorphisms of D^n onto itself. We show here that, under a suitable topology, the injection $\mathrm{SO}(n)\to\mathrm{Diff}(D^n)$ is a weak homotopy equivalence. If follows as a corollary that every orientation preserving diffeomorphism of S^n onto itself which extends to a diffeomorphism of D^n is isotopic to the identity through such diffeomorphisms. This partially answers a question of Smale.

In the last section of the paper, we consider $\mathrm{Diff}(D^n)$ in the C^1 topology and show that either $\mathrm{SO}(6)\to\mathrm{Diff}(D^6)$ is not a weak homotopy equivalence or $\mathrm{SO}(6)$ is not a deformation retract of $\mathrm{Diff}(S^5)$.

2. Preliminaries. Suppose $f\in\mathrm{Diff}(D^n)$, $\varepsilon>0$, and C is a compact subset of the interior of D^n. Let $W(f,\varepsilon,C)$ denote the set of all $g\in\mathrm{Diff}(D^n)$ such that

$$|f(x)-g(x)|<\varepsilon \qquad \text{for all } x\in D^n$$

and

$$|\partial f_i/\partial x_k(x)-\partial g_i/\partial x_k(x)|<\varepsilon \qquad \text{for all } x\in C;\ i,k=1,\cdots,n.$$

We take the sets $W(f,\varepsilon,C)$ as a basis for our special topology on $\mathrm{Diff}(D^n)$.

Let B^n denote the interior of D^n and let $\mathrm{Diff}(B^n)$ denote the group of orientation preserving homeomorphisms of B^n in the coarse C^1 topology [6]. Let $\mathrm{EDiff}(B^n)$ denote the subset of $\mathrm{Diff}(B^n)$ consisting of elements which are extendable to diffeomorphisms of D^n. We endow $\mathrm{EDiff}(B^n)$ with the topology it inherits from $\mathrm{Diff}(B^n)$. We let $\mathrm{EDiff}(D^n)$ denote the set $\mathrm{Diff}(D^n)$ with the topology induced from $\mathrm{EDiff}(B^n)$ by the inclusion map $i: B^n\to D^n$.

Stewart [9] has shown that $\mathrm{SO}(n)$ is a strong deformation retract of $\mathrm{Diff}(B^n)$. Since $\mathrm{EDiff}(B^n)$ is mapped into itself throughout this deformation retraction, we have that $\mathrm{SO}(n)$ is a strong deformation retract of $\mathrm{EDiff}(B^n)$ also.

Let $\mathrm{EDiff}(S^n)$ denote the set of orientation preserving diffeomorphisms of S^n onto itself which are extendable to diffeomorphisms of D^n. We give $\mathrm{EDiff}(S^n)$ the compact-open topology.

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3. Diff (D^n) in the special topology.

Theorem 1. *The injection $SO(n) \to Diff(D^n)$ is a weak homotopy equivalence.*

Proof. Stewart [9] showed that $SO(n)$ is a strong deformation retract of Diff (R^n) in the coarse C^1 topology. It follows that $SO(n) \to$ Diff(D^n) is a weak homotopy equivalence if and only if every R^n bundle over a finite complex with group Diff (R^n) contains a unique D^n bundle with group Diff (D^n) (Cf. [1]). By the result of Stewart just quoted, every R^n bundle over a finite complex with group Diff (R^n) contains a unique D^n bundle with group $SO(n)$. To complete the proof, it is sufficient to show that every D^n bundle over a finite complex with group Diff (D^n) is equivalent to a D^n bundle with group $SO(n)$.

Let ξ be a D^n bundle over a finite complex with group Diff (D^n). Let ξ' be the EDiff (D^n) bundle induced by ξ. Then ξ' is equivalent to a D^n bundle η with group $SO(n)$ since EDiff (D^n) is homotopy equivalent to $SO(n)$. It follows that ξ and η are equivalent as Ehresmann-Feldbau bundles. Since the topology we have put on Diff (D^n) is the *union* [10, p. 131] of the EDiff and compact-open topologies, it follows by § 5 of [8] that ξ and η are equivalent and the proof is complete.

Corollary. EDiff (S^n) is pathwise connected.

In problem 21 of the Seattle conference notes [4], Smale asks whether EDiff (S^n) is pathwise connected in the C^1 topology. Cerf has announced an affirmative answer to this question for $n \geq 8$ [3].

4. Diff (D^6) in the C^1 topology.

Theorem 2. *Either the injection $SO(6) \to Diff(D^6)$ is not a weak homotopy equivalence or $SO(6)$ is not a deformation retract of $Diff(S^5)$.*

Proof. Let J: Diff $(S^5) \times I \to$ Diff (S^5) be a deformation retraction of Diff (S^5) onto $SO(6)$, where I denotes the closed unit interval $[0, 1]$, and suppose $f \in$ Diff (S^5). Define $\bar{f}(\theta, t) = tJ_{1-t}(\theta)$ where (θ, t) are polar coordinates for D^n (i.e., $\theta \in S^{n-1}, t \in I$). It is easy to see that $\bar{f} \in$ Diff (D^n). It follows by this construction that Diff (D^6) is the topological product Diff $(S^5) \times$ Diff $(D^6; S^5)$ where Diff $(D^6; S^5)$ is the subgroup of Diff (D^6) consisting of diffeomorphisms which are the identity on S^5 in the C^1 topology and Diff (S^5) is the group of orientation preserving homeomorphisms of S^5 onto itself in the C^1 topology.

If $SO(6) \to$ Diff (D^6) is a weak homotopy equivalence, then $\pi_i($Diff $(D^6; S^5)) = 0$ for all i. Cerf [2] has shown that $\pi_i($Diff $(S^n))$ is isomorphic to the direct sum $\pi_i(SO(n+1)) + \pi_i($Diff $(D^n; S^{n-1}))$ for all n and all i. Milnor [5] has shown that $\pi_0(\mathrm{Diff}\,(S^6))\neq 0$. Thus $\pi_0(\mathrm{Diff}\,(D^6; S^5))\neq 0$ and we have a contradiction.

Smale [7] showed that $\mathrm{SO}\,(n+1)$ is a deformation retract of $\mathrm{Diff}\,(S^n)$ for $n\leq 2$. It follows by Milnor's work [5] that this is generally false for $n\geq 6$.

It follows from Smale's work [7] that the injection $\mathrm{SO}\,(n)\rightarrow \mathrm{Diff}\,(D^n)$ is a weak homotopy equivalence for $n\leq 2$. Cerf [2] has shown that $\pi_0(\mathrm{Diff}\,(D^3))=0$. There are no further results known on the homotopy of $\mathrm{Diff}\,(D^n)$ in the C^1 topology.

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