95 *A Note on the Analyticity in Time and the Unique Continuation Property for Solutions of Diffusion Equations*

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1. **Introduction.** Consider the equation of evolution in $L^2(G)$

(1) $$du/dt = Au, \qquad t>0,$$

where G is a domain in R^n. We assume that A is an infinitesimal generator of holomorphic semi-groups $S(t)$ with the domain $\boldsymbol{D}(A)$ of $A \supset C_0^\infty(G)$, and that $A\varphi = \sum_{i,j=1}^n a_{ij}(x)\partial^2\varphi/\partial x_i \partial x_j + \sum_{j=1}^n a_j(x)\partial\varphi/\partial x_j + a(x)\varphi(\equiv A\varphi)$ for $\varphi \in C_0^\infty(G)$ where the coefficients satisfy the following conditions: $a_{ij}(x)$ are functions of class C^2 and with second derivatives locally Hölder continuous, i.e., $a_{ij}(x) \in C_{loc}^{2+h}(G)(0<h<1)$, $a_j(x)$ are of C^1, and $a(x)$ of $C_{loc}^h(G)$; the matrix $\{a_{ij}(x)\}$ is positive definite everywhere in G. The purpose of this note is to show the following theorems.

Theorem 1. *For any $f \in L^2(G)$, there exists a function $u(x, t)$ in $C_{loc}^{2+h}(G\times(0, \infty)$ such that for any fixed $t>0$ $u(x, t)=S(t)f(x)$ after a correction of a null set of the space R^n. Moreover, for any fixed x in G, $u(x, t)$ is analytic in t.*

Theorem 2. *Let f be in $L^2(G)$. If for a fixed $t_0>0$, $S(t_0)f(x)=0$ for almost every x in some nonempty open subset U of G, then $S(t)f$ vanishes identically in $G\times(0, \infty)$.*

The regularity of semi-group solutions of the diffusion equations was studied by K. Yosida [1] H. Komatsu [2], and others, under somewhat strong conditions on the coefficients. The unique continuation property of solutions of the diffusion equations was studied by Itô-Yamabe [3], Mizohata [4], Yosida [1], Shirota [5], and others. The proof of Theorems 1 and 2, shown in the next section, is suggested by K. Yosida [1]. We can extend our results in some directions:

1°. Instead of Eq. (1), we can consider the equation $du/dt = A(t)u$, where $A(t)$ are generators of holomorphic semi-groups satisfying certain conditions.

2°. The condition that the restriction of A on $C_0^\infty(G)$ is an elliptic operator of second order can be weakened to the following one; It is an elliptic operator of order 2 m with smooth coefficients such that an operator $A+(\partial^2/\partial\xi^2+\partial^2/\partial\eta^2)^m-\partial/\partial\xi$ on $G\times R^2$ has a unique continuation property.

2. **Proof of Theorem 1.** Since $S(t)$ is a holomorphic semi-group, $S(t)f$ admits a holomorphic extension $S(z)f$ given by strongly convergent Taylor series:

(2) $$S(z)f=\sum_{m=0}^{\infty}(z-t)^m S^{(m)}(t)f/m!$$

for z in the sector $\sum=\{z;\,|\arg z|<\theta\}$, $S^{(m)}(t)f$ being the m-th derivative in t of $S(t)f$. Furthermore,

(3) $S^{(m)}(t)f\in \boldsymbol{D}(\boldsymbol{A})$, and $\boldsymbol{A}S^{(m)}(t)f=S^{(m+1)}(t)f$, $t>0$, $m=0,1,\cdots$.

Since $(S(z)f, S(z)f)$ is continuous for z in $\sum$, setting $v(x,\xi,\eta)=S(\xi+i\eta)f$, we see that $v(x,\xi,\eta)$ is square-integrable over any compact set in $G\times C$, where $(\cdot,\cdot)$ is the scalar product of $L^2(G)$ and $C=\{(\xi,\eta);\ \xi+i\eta\in\sum\}$. We shall show that $v(x,\xi,\eta)$ satisfies the equation

(4) $$(v[\partial^2/\partial\xi^2+\partial^2/\partial\eta^2+\partial/\partial\xi]\Phi+A'\Phi)_{L^2(Z)}=0$$

for any Φ in $C_0^\infty(Z)$, where $Z=G\times C$ and $(\cdot,\cdot)_{L^2(Z)}$ is the scalar product of $L^2(Z)$. For any φ in $C_0^\infty(G)$, $(v(\cdot,\xi,\eta),\varphi)$ is a harmonic function of ξ and η, since $(S(z)f,\varphi)$ is holomorphic in Σ. Hence,

(5) $$(v,[\partial^2/\partial\xi^2+\partial^2/\partial\eta^2]\varphi\psi)_{L^2(Z)}=0$$

for any φ in $C_0^\infty(G)$ and any ψ in $C_0^\infty(C)$. On the other hand, by (2) and (3), $S(\xi+i\eta)f$ satisfies the equation $\dfrac{\partial(S(\xi+i\eta)f,\varphi)}{\partial\xi}=(\boldsymbol{A}S(\xi+i\eta)f,\varphi)$ for any (ξ,η) in C and any φ in $C_0^\infty(G)$. Hence,

(6) $$(v,(\partial/\partial\xi+A')\varphi\psi)_{L^2(Z)}=0$$

for φ in $C_0^\infty(G)$ and ψ in $C_0^\infty(C)$, where A' is the formal adjoint of A. Since the totality of finite sums $\sum\varphi_j\psi_j$ with $\varphi_j\in C_0^\infty(G)$ and $\psi_j\in C_0^\infty(C)$, is dense in $C_0^\infty(Z)$ in the topology of $\mathcal{D}(Z)$ (see L. Schwartz [6]), we have, by (5) and (6),

(7) $$(v,[\partial^2/\partial\xi^2+\partial^2/\partial\eta^2]\Phi)_{L^2(Z)}=0$$

and

(8) $$(v,[\partial/\partial\xi+A']\Phi)_{L^2(Z)}=0$$

for any Φ in $C_0^\infty(Z)$. Hence, by adding (7) to (8), we see that Eq. (4) holds. Since $\partial^2/\partial\xi^2+\partial^2/\partial\eta^2+A-\partial/\partial\xi$ is an elliptic differential operator, applying the theorem on the interior regularity for weak solutions of elliptic equations (see F. Browder [7] p. 129), we see, by (4) and (8), that, after a correction of a null set of the product space $R^n\times R^2$, $v(x,\xi,\eta)$ is equal to a function $v^*(x,\xi,\eta)\in C_{loc}^{2+h}(Z)$ which satisfies the equation

(9) $$\partial v^*(x,\xi,\eta)/\partial\xi=Av^*(x,\xi,\eta)\ \text{in}\ Z.$$

Furthermore, we have, by (7),

(10) $$[\partial^2/\partial\xi^2+\partial^2/\partial\eta^2]\,v^*(x,\xi,\eta)=0\ \text{in}\ Z.$$

Hence, for any fixed x in G, $v^*(x,\xi,\eta)$ is an analytic function of ξ and η. Hence, we see that $v^*(x,t,0)$ is a function in $C_{loc}^{2+h}(G\times(0,\infty))$ which satisfies Eq. (9) in $G\times(0,\infty)$, and that for any fixed

x in G, $v^*(x, t, 0)$ is analytic in t. Since for φ in $C_0^\infty(G)$, $(v(\cdot, \xi, \eta), \varphi)$ is continuous in ξ and η, $(v(\cdot, \xi, \eta), \varphi)=(v^*(\cdot, \xi, \eta), \varphi)$ for any (ξ, η) in C. Hence, for any fixed $t>0$, $S(t)f(x)$ is equal to a function $v^*(x, t, 0)$, after a correction on a null set of the space R^n. Thus, Theorem 1 is proved.

3. Proof of Theorem 2. We first show that

(11) $\quad (S^{(n)}(t_0)f, \varphi)=0$ for φ in $C_0^\infty(U)$ and $n=0, 1, 2, \cdots$.

The assumption of Theorem 2 implies that (11) holds for $n=0$. Suppose that $(S^{(k)}(t_0)f, \varphi)=0$ for φ in $C_0^\infty(U)$. Then we have, by (3), $(S^{(k+1)}(t_0)f, \varphi)=(AS^{(k)}(t_0)f, \varphi)=(S^{(k)}(t_0)f, A'\varphi)=0$, showing that $(S^{(k+1)}(t_0)f, \varphi)=0$, for φ in $C_0^\infty(U)$. Thus we have (11). Since $S(z)f$ is holomorphic in Σ, setting $v(x, \xi, \eta)=S(\xi+i\eta)f$, we have, by (2) and (11), $v(x, \xi, \eta)=0$ for almost every (x, ξ, η) in $U\times C$, so that $v^*(x, \xi, \eta)=0$ for any (x, ξ, η) in $U\times C$, where v^* is defined in the proof of Theorem 1. Since $v^*(x, \xi, \eta)$ satisfies Eq. (4), applying the unique continuation theorem for solutions of elliptic differential equations of second order, we have $v^*(x, \xi, \eta)=0$ in Z. Hence, $u(x, t)=v^*(x, t, 0)=0$ for any (x, t) in $G\times(0, \infty)$. Thus Theorem 2 is proved.

References

[1] Yosida, K.: An abstract analyticity in time for solutions of a diffusion equation. Proc. Japan Acad., **35**, 109-113 (1959).

[2] Komatsu, H.: A characterization of real analytic functions. Proc. Japan Acad., **36**, 90-93 (1960).

[3] Ito, S., and Yamabe, H.: A Unique continuation theorem for solutions of a parabolic differential equation. J. Math. Soc. Japan, **10**, 314-321 (1958).

[4] Mizohata, S.: Unicité du prolongement des solutions pour quelques opérateurs differentiels paraboliques. Mém. Coll. Sci. Univ. Kyoto, Ser. A, **31**, 219-239 (1958).

[5] Shirota, T.: A unique continuation theorem of a parabolic differential equation. Proc. Japan Acad., **35**, 455-460 (1959).

[6] Schwartz, L.: Théorie des Distributions, vol. 1. Hermann, Paris (1950).

[7] Browder, F.: Functional analysis and partial differential equations. II. Math. Ann., **145**, 81-226 (1962).