143. *On the Cauchy Problem for the Equation with Multiple Characteristic Roots*

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1. Introduction. 1.1. S. Mizohata [1] obtained the necessary condition for the well posedness in Petrowsky's sense of the Cauchy probem for

$$M[u]=\frac{\partial}{\partial t}u-\sum_{j=1}^{n}A_j(x,t)\frac{\partial}{\partial x_j}u$$

where $\{A_j(x,t)\}$ are $N\times N$ matrices which are bounded and sufficiently smooth in x and t.

In [1] the first approximation to M plays an important part. M is approximated by the singular integral operator associated with tangential operator.

Now we consider the higher order approximation to differential operator in some sense, and get a result presented in the following paragraphs.

1.2. Consider the differential operator

$$L=\left(\frac{\partial}{\partial t}\right)^m+\sum_{\substack{|\nu|+j\leqq m\\ j\leqq m-1}}a_{\nu,j}(x,t)\left(\frac{\partial}{\partial x}\right)^{\nu}\left(\frac{\partial}{\partial t}\right)^{j} \tag{1}$$

where

$$x=(x_1,\cdots,x_n),\qquad \left(\frac{\partial}{\partial x}\right)^{\nu}=\left(\frac{\partial}{\partial x_1}\right)^{\nu_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\nu_n}$$

and $\{a_{\nu,j}(x,t)\}$ are contained in $\mathscr{B}_{x,t}$.

We denote the principal part of L by

$$L_0=\left(\frac{\partial}{\partial t}\right)^m+\sum_{\substack{|\nu|+j=m\\ j\leqq m-1}}a_{\nu,j}(x,t)\left(\frac{\partial}{\partial x}\right)^{\nu}\left(\frac{\partial}{\partial t}\right)^{j} \tag{2}$$

and associate the characteristic equation to it:

$$L_0(x,t,\xi;\lambda)=\lambda^m+\sum_{\substack{|\nu|+j=m\\ j\leqq m-1}}a_{\nu,j}(x,t)\xi^{\nu}\lambda^{j}=0 \tag{3}$$

where $\xi^{\nu}=\xi_1^{\nu_1}\cdots\xi_n^{\nu_n}$.

1.3. We consider the Cauchy problem for (1) in L^2 sense.

Definition. *The Cauchy problem for* (1) *is said to be well posed in* L^2 *sense if there exists a unique solution* $u=u(x,t)$ *of* $Lu=0$ *such that*

$$u(x,t)\in\mathcal{E}_t^0(\mathcal{D}_{L^2}^{m-1})\cap\cdots\cap\mathcal{E}_t^{m-1}(L^2),\ (0\leqq t\leqq T) \tag{4}$$

for any initial data Ψ

$$(5) \qquad \Psi=\left\{\left(\frac{\partial}{\partial t}\right)^{j} u\Big|_{t=0}=u_j(x)\in\mathcal{D}_{L^2}^{m-j-1},\ j=0, 1, \cdots, m-1\right\}.$$

Our result is

Theorem. *If* (3) *has multiple characteristic roots with constant multiplicity, then the Cauchy problem for* (1) *is not well posed in* L^2 *sense.*

1.4. Our theorem means essentially the following fact: If (3) has multiple characteristic roots with constant multiplicity, then there exists a lower order operator B for L_0, such that the Cauchy problem for $(L_0+B)u=0$ is not well posed in L^2 sense. In fact, if there exists such a B we decompose L which has L_0 as its principal part as follows:

$$(6) \qquad L=L_0+B+\{(L-L_0)-B\}.$$

Then we can prove that the Cauchy problem for (6) is not well posed in L^2 sense with the same reasoning as for L_0+B. Because $\{(L-L_0)-B\}$ is a lower order differential operator.

1.5. We shall prove our theorem only when L_0 has a double characteristic root, the general case can be treated by the same fashion. First we formulate the following two conditions (I) and (II) about L_0:

(I) All roots of (3) are real for any real $\xi\neq 0$.

(II) There exist a neighbourhood Ω_0 of $(x, t)=(0, 0)$ and a neighbourhood Ω_1 of $\xi_0'=\xi_0/|\xi_0|$ on the unit sphere such that for all $(x, t, \xi)\in\Omega_0\times\Omega_1$, $L_0(x, t, \xi; \lambda)$ can be written as

$$L_0(x, t, \xi; \lambda)=(\lambda-\lambda_1)^2\prod_{j\neq 1}(\lambda-\lambda_j)$$

where $\{\lambda_j\}_{j\neq 1}$ are distinct roots of (3). Then we have

Lemma. *Assume that* (2) *satisfies* (I) *and* (II). *Then there exists a differential operator B of lower order such that the Cauchy problem for*

$$(7) \qquad (L_0+B)u=0$$

is not well posed in L^2 *sense.*

The proof of this Lemma is given in the paragraph 4 and get our *Theorem* as remarked above.

2. Approximation to L_0+B. **2.1.** Defining the lower order operator B by for the case: $\xi_0'=(1, 0, \cdots, 0)$

$$(8) \qquad B=b\left(\frac{\partial}{\partial x_1}\right)^{m-1},\ b\text{: real constant to be determined later,}$$

we can write (7) in the following system with a new unknown vector $U={}^t\left(u, \left(\frac{\partial}{\partial t}\right)u, \cdots, \left(\frac{\partial}{\partial t}\right)^{m-1}u\right)$:

$$(9) \qquad \frac{\partial}{\partial t}U=A\left(x, t, \frac{\partial}{\partial x}\right)U$$

where

$$(10)\quad A\left(x, t, \frac{\partial}{\partial x}\right)=\begin{bmatrix} 0, & 1, & 0, \cdots, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot\ 0, & \cdot\ 1 \\ -a_m\left(x, t, \frac{\partial}{\partial x}\right)-b\left(\frac{\partial}{\partial x_1}\right)^{m-1}, & \cdots, & & -a_1\left(x, t, \frac{\partial}{\partial x}\right) \end{bmatrix}$$

$$a_j\left(x, t, \frac{\partial}{\partial x}\right)=\sum_{|\nu|=j} a_{\nu, m-j}(x, t)\left(\frac{\partial}{\partial x}\right)^{\nu}.$$

2.2. Take functions $\beta(x)\in C_x^\infty$ and $\hat{\alpha}(\xi)\in C_\xi^\infty$ with small supports, which take the value 1 in a neighbourhood of $x=0$ and in a neighbourhood of ξ_0 (in which $\xi=0$ is not contained), respectively.

Defining $\hat{\alpha}_n(\xi)$ by $\hat{\alpha}_n(\xi)=\hat{\alpha}\left(\frac{\xi}{n}\right)$, we denote the Fourier inverse image of $\hat{\alpha}_n(\xi)$ by $\alpha_n(x)$. Then $\alpha_n(x)$ is analytic.

First we multiply (9) by $\beta(x)$. Next we apply the convolution operator $\alpha_n(x)*$. Then we get

$$(11)\quad \frac{\partial}{\partial t}\alpha_n*(\beta U)=A\left(x, t, \frac{\partial}{\partial x}\right)(\alpha_n*(\beta U)) + [\alpha_n*, A](\beta U) + \alpha_n*([\beta, A]U).$$

Take the operator

$$E_m(\Lambda)=\begin{bmatrix} \{i(\Lambda+1)\}^{m-1} & & & & \\ & \{i(\Lambda+1)\}^{m-2} & & & 0 \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & \cdot & 1 \end{bmatrix},$$

and apply to (11). Then we get

$$(12)\quad \frac{\partial}{\partial t}E_m\alpha_n*(\beta U)=E_mAE_m^{-1}(E_m\alpha_n*(\beta U)) + [\alpha_n*, AE_m^{-1}]E_m(\beta U)+\alpha_n*([\beta, AE_m^{-1}]E_mU).$$

It is not hard to see that $[\alpha_n*, AE_m^{-1}]$ and $[\beta, AE_m^{-1}]$ are bounded operators in L^2.

2.3. We can approximate $E_mAE_m^{-1}$ by the singular integral operator $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1$ modulo bounded operators in L^2:

$$(13)\quad E_m(\Lambda)A\left(x, t, \frac{\partial}{\partial x}\right)E_m^{-1}(\Lambda)=(\mathcal{H}_0+\mathcal{H}_1)\Lambda+B_1$$

where

$$(14)\quad \mathcal{H}_0=\begin{bmatrix} 0, & i, & \cdots, & 0 \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ 0, & \cdots, & 0, & i \\ h_m, & \cdots, & & h_1 \end{bmatrix}, \quad \mathcal{H}_1=\begin{bmatrix} & & 0 & \\ & & & \\ & & & \\ b_0, & 0, & \cdots, & 0 \end{bmatrix}$$

with the symbols

$$(15)\qquad \sigma(h_j) = -i \sum_{|\nu|=j} a_{\nu,m-j}(x, t)\hat{\gamma}(\xi)\frac{\xi^\nu}{|\xi|^j}$$

$$\sigma(b_0) = ib\hat{\gamma}(\xi)\frac{\xi_1^{m-1}}{|\xi|^m}.$$

B_1 is a bounded operator in L^2. Finally $\hat{\gamma}(\xi)$ is a function which is infinitely differentiable, and vanishes for $|\xi| \leqq R(>1)$ and takes the value 1 for $|\xi| \geqq R+1$ as $0 \leqq \hat{\gamma}(\xi) \leqq 1$.

Now we set $V_n = E_m(\Lambda)\alpha_n * (\beta U)$ and $F_n = [\alpha_n *, AE_m^{-1}]E_m(\beta U) + \alpha_n * ([\beta, AE_m^{-1}]E_m U)$. Using (13), we get from (12)

$$(16)\qquad \frac{d}{dt}V_n = (\mathcal{H}_0 + \mathcal{H}_1)\Lambda V_n + B_1 V_n + F_n.$$

3. Differential inequality. 3.1. First we shall calculate the eigenvalues of $\sigma(\mathcal{H}) = \sigma(\mathcal{H}_0) + \sigma(\mathcal{H}_1)$. We set $A_0 = \sigma(\mathcal{H}_0)$ and $A_1 = \sigma(\mathcal{H}_1)|\xi|$. Following to the method due to Vishik-Lyusternik [2], we can get the eigenvalues of $A_\varepsilon = A_0 + \varepsilon A_1$ ($\varepsilon = 1/|\xi|$) as the perturbation to the eigenvalues of A_0.

Considering the condition (II) about L_0, the eigenvalues of A_ε are given in the following Puiseux expansion form for sufficiently small ε:

$$(17)\qquad \begin{aligned} \lambda_{\varepsilon,1} &= \lambda_1 + \lambda_1^{(1)}\varepsilon^{1/2} + \lambda_2^{(1)}\varepsilon + \cdots \\ \lambda_{\varepsilon,2} &= \lambda_1 + \lambda_1^{(2)}\varepsilon^{1/2} + \lambda_2^{(2)}\varepsilon + \cdots \\ \lambda_{\varepsilon,3} &= \lambda_2 + \lambda_1^{(3)}\varepsilon + \lambda_2^{(3)}\varepsilon^2 + \cdots \\ &\vdots \\ \lambda_{\varepsilon,m} &= \lambda_{m-1} + \lambda_1^{(m)}\varepsilon + \lambda_2^{(m)}\varepsilon^2 + \cdots \end{aligned}$$

3.2. Taking the method for getting the coefficients of these expansions into account, $\{\lambda_{\varepsilon,j}(x, t, i\xi)\}$ are sufficiently smooth to be the symbols of singular integral operators. We consider singular integral operators $R_{\varepsilon,1}, \cdots, R_{\varepsilon,m}$ defined by the symbols $\hat{\gamma}(\xi)\lambda_{\varepsilon,1}, \cdots, \hat{\gamma}(\xi)\lambda_{\varepsilon,m}$ respectively.

3.3. Taking b conveniently, there exists a positive constant c_1 such that

$$(18)\qquad \operatorname{Re}\lambda_1^{(1)} \geqq c_1 \quad \text{and} \quad \operatorname{Re}\lambda_1^{(2)} \leqq -c_1.$$

3.4. Denote the Vandermonde matrix with respect to $\{\lambda_{\varepsilon,j}\}_{j=1}^m$ by $\sigma(\mathcal{N}_1)$. Define $\sigma(\mathcal{N})$ by

$$\sigma(\mathcal{N}) = \hat{\gamma}(\xi)|\xi|^{-1/2}E\cdot\sigma(\mathcal{N}_1)^{-1}$$

where E is the $m \times m$ unit matrix. Then $\sigma(\mathcal{N})$ defines a singular integral operator $\mathcal{N}$ which diagonalize $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ into

$$\mathcal{D} = \begin{bmatrix} R_{\varepsilon,1} & & 0 \\ & \ddots & \\ 0, & & R_{\varepsilon,m} \end{bmatrix}$$

modulo bounded operators in L^2: $\sigma(\mathcal{N})\sigma(\mathcal{H}) = \sigma(\mathcal{D})\sigma(\mathcal{N})$. Using $\equiv$ to denote equalities modulo bounded operators in L^2, we get

$$\mathcal{N}\mathcal{H}\Lambda \equiv \mathcal{N}\circ\mathcal{H}\Lambda = \mathcal{D}\circ\mathcal{N}\Lambda \equiv \mathcal{D}\mathcal{N}\Lambda \equiv \mathcal{D}\Lambda\mathcal{N}$$

where $A\circ B$ means a singular integral operator whose symbol is $\sigma(A)\cdot\sigma(B)$. Then setting $W_n = \mathcal{N}V_n$, we get from (16) after the operation of $\mathcal{N}$

$$\frac{d}{dt}W_n = \mathcal{D}\Lambda W_n + \mathcal{N}'V_n + B_2V_n + \mathcal{N}B_1V_n + \mathcal{N}F_n$$

where $\mathcal{N}'$ is the singular integral operator with the symbol $\sigma(\mathcal{N}') = -\frac{d}{dt}\sigma(\mathcal{N})$.

3.5. Taking a positive constant K, we define $S_n(t)$ by

$$S_n(t) = K\,\|W_n^{(1)}(t)\|_{L^2}^2 - \sum_{j=2}^{m}\|W_n^{(j)}(t)\|_{L^2}^2.$$

We shall define the size of K later.

Now we can prove that $S_n(t)$ satisfies the following differential inequality:

$$\frac{d}{dt}S_n(t) \geqq c_1\sqrt{n}\,S_n(t) - c_2\|V_n\|_{L^2}^2 - c_3\|F_n\|_{L^2}^2, \tag{19}$$

where c_1, c_2, and c_3 are constants independent of n. In fact, setting $G_n = B_2V_n + \mathcal{N}B_1V_n + \mathcal{N}F_n + \mathcal{N}'V_n$, we get

$$\frac{d}{dt}S_n(t) = 2K\,\mathrm{Re}\,(R_{\varepsilon,1}W_n^{(1)}, W_n^{(1)}) + 2K\,\mathrm{Re}\,(G_n^{(1)}, W_n^{(1)})$$
$$-2\sum_{j=2}^{m}\mathrm{Re}\,(R_{\varepsilon,j}W_n^{(j)}, W_n^{(j)}) - 2\sum_{j=2}^{m}\mathrm{Re}\,(G_n^{(j)}, W_n^{(j)}).$$

From this, (19) follows by (18) and Plancherel's equality.

4. **Proof of Lemma.** 4.1. We shall prove *Lemma* by a contradiction. (1°) First we assume that the Cauchy problem for (7) is well posed in L^2 sense. Then the energy inequality holds:

$$E(t\,;u) \leqq CE(o:u) \tag{20}$$

where

$$E(t:u) = \sum_{j=0}^{m-1}\left\|\left(\frac{\partial}{\partial t}\right)^j u(t)\right\|_{m-j-1}.$$

(2°) On the other hand, if the Cauchy problem for (7) with any initial data (5) has a solution (4) for arbitrary lower order term B, then taking B conveniently we can show that for any positive constant C there exists a solution of (7) which does not satisfy the energy inequality (20).

(1°) and (2°) are just contradictory consequences. (1°) is a simple consequence of Banach's closed graph theorem, therefore we only have to show (2°) to get our *Lemma*.

4.2. Now we shall show (2°). Let $\hat{\psi}(\xi) \in C_\xi^\infty$ be a function with a compact support and take the value 1 on the support of $\hat{\alpha}(\xi)$. Defining $\hat{\psi}_n(\xi)$ by $\hat{\psi}_n(\xi) = \hat{\psi}(\xi - (n-1)\xi_0)$, we denote the Fourier inverse image of $\hat{\psi}_n(\xi)$ by $\psi_n(x)$.

Using B defined in 3.3, we shall consider the Cauchy problem

$$\text{(21)}\qquad \begin{cases}(L_0+B)u=0\\ u(o)=\cdots=\left(\dfrac{\partial}{\partial t}\right)^{m-2}u\,|_{t=0}=0, \quad \left(\dfrac{\partial}{\partial t}\right)^{m-1}u|_{t=0}=\psi_n(x)\end{cases}$$

in L^2 sense. Denote the solution of (21) by $u_n(x, t)$:

$$u_n(x, t)\in\mathcal{E}_t^0(\mathcal{D}_{L^2}^{m-1})\cap\cdots\cap\mathcal{E}_t^{m-1}(L^2), \qquad 0\leqq t\leqq T.$$

Replacing U in (9) by $U_n={}^t\!\left(u_n, \cdots, \left(\dfrac{\partial}{\partial t}\right)^{m-1}u_n\right)$, the same reasoning as in the paragraph 3 guarantee (19) for U_n.

Now we assume that $u_n(x, t)$ satisfies the energy inequality (20). Then it follows that

$$\text{(22)}\qquad \|V_n\|\leqq C, \ \|F_n\|\leqq C', \quad \text{and} \quad S_n(t)\leqq C''$$

where C, C', and C'' are constants independent of n. Using (22) we get from (19)

$$\text{(23)}\qquad \frac{d}{dt}S_n(t)\geqq c_1\sqrt{n}\,S_n(t)-c_2$$

where c_1 and c_2 are constants independent of n. Integrating (23) by t and taking the last term of (22) into account, we get

$$C\geqq S_n(t)\geqq e^{c_1\sqrt{n}\,t}S_n(o)+\frac{c_2}{c_1\sqrt{n}}(1-e^{c_1\sqrt{n}\,t}).$$

In addition, taking K suitably we can prove that

$$\text{(24)}\qquad S_n(o)\geqq c>o$$

holds for some constant c. In the sequel

$$C\geqq S_n(t)\geqq ce^{c_1\sqrt{n}\,t}+\frac{c_2'}{c_1\sqrt{n}}(1-e^{c_1\sqrt{n}\,t}).$$

This is an apparent contradiction as n tends to infinity, and (2°) is proved.

References

[1] Mizohata, S.: Some remarks on the Cauchy problem. J. Math. Kyoto Univ., **1**, 109-127 (1961).

[2] Vishik, M. I., and Lyusternik, L. A.: The solution of some perturbation problem for matrices and selfadjoint or non-selfadjoint differential equations. I., Russian Math. Survey (1960).