# 142. Eigenfunction Expansions Associated with the Schrödinger Operator with a Complex Potential and the Scattering Inverse Problem

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**1. Introduction.** In this note[1] we are concerned with the Schrödinger operator $-\Delta+q(x)$ acting in the Hilbert space $\mathfrak{H}=L^2(E_3)$, where $E_3$ denotes the 3-dimensional Euclidean space. We consider the case where $q(x)$ is a complex-valued potential function assumed to satisfy the following conditions:

(A) $q(x)\in L^2(E_3)$, is locally Hölder continuous except for a finite number of singularities and behaves like $O(|x|^{-2-\delta})(\delta>0)$ as $|x|\to\infty$.

The eigenfunction expansion theorem associated with $-\Delta+q(x)$ was already proved, based on a work of Povzner [7], by Ikebe [1] under the same assumptions on $q(x)$ when it is real-valued. Our purpose is to extend his results to the case of complex-valued potentials. We use the methods developed by J. Schwartz [8], Kato [3], and Kuroda [4], [5], and follow almost the same line of the proof given by Ikebe. In our case, however, the existence of a uniformly bounded spectral resolution $E(e)$ of $-\Delta+q(x)$ is not proved if we choose real intervals $e$ arbitrarily. So our results on the expansion problem will become rather of a local character.

The expansion formula can be applied to solve the scattering inverse problem formulated by Faddeev in [2]. His result is the following: A real-valued potential function $q(x)$ can be determined uniquely, under the assumptions that $q(x)\in C^1(E_3)$ and

$$(\mathrm{A}_1)\qquad q(x)=O(|x|^{-3-\delta})(\delta>0) \text{ as } |x|\to\infty,$$

from the assymptotic conditions for $|k|\to\infty$ of the function $\theta_\pm(n,\nu;|k|)$ having a physical meaning.[2] We shall extend this result also to the case of complex-valued potential assumed to satisfy $(\mathrm{A}_1)$ in addition to (A). In our proof it is not necessary to assume $q(x)\in C^1(E_3)$.

**2. Spectral resolutions.** We consider $-\Delta+q(x)$ to be defined on $C_0^\infty(E_3)$. We denote by $L_0$ the selfadjoint extension of $-\Delta$ with

1) The detailed proof of the following results will be given in a forthcoming paper.

2) $|\theta_-|^2$ gives the so-called differential closs section of scattering for the particle incident in the direction $\nu$ and scattered in the direction $n$. For the definition of $\theta_\pm(n,\nu;|k|)$ see (25) in § 4.

the domain $\mathfrak{D}(L_0)=\mathfrak{D}^2_{L^2}$ and by $V$ the multiplicative operator given by $q(x)$. Put

(1) $$L=L_0+V, \qquad \mathfrak{D}(L)=\mathfrak{D}(L_0).$$

Then $L$ defines a unique closed extension of $-\Delta+q(x)$. The adjoint operator $L^*$ of $L$ is given by $L^*=L_0+V^*$, $\mathfrak{D}(L^*)=\mathfrak{D}(L)$, where $V^*$ represents the operator of multiplication by the complex conjugate $\overline{q(x)}$ of $q(x)$. The essential spectrum of $L$ is composed of the value $\mu=0$ and the real interval $(0, \infty)$ which is the continuous spectrum of $L$, and a value $\mu \notin [0, \infty)$ is a discrete eigenvalue of $L$ if and only if $\bar{\mu}$ is a discrete eigenvalue of $L^*$ (Cf. [6]; Theorem 1.1).

We denote by $R_0(\zeta)$, $R(\zeta)$, and $R^*(\zeta)$ the resolvents of $L_0$, $L$, and $L^*$, respectively. In virtue of (1) we have

(2) $$R(\zeta)=R_0(\zeta)-R_0(\zeta)VR(\zeta), \qquad R(\zeta)^*=R^*(\bar{\zeta}),$$

where by $R(\zeta)^*$ we denote the adjoint operator of $R(\zeta)$. Remark that $R_0(\zeta)$ is the integral operator generated by the kernel $(4\pi\,|x-y|)^{-1}\exp\{i\sqrt{\zeta}\,|x-y|\}$, where by $\sqrt{\zeta}$ is meant the branch of the square root of $\zeta$ with $\operatorname{Im}\sqrt{\zeta}>0$. It is natural to define the spectral resolutions $E(e)$ of $L$, for a real interval $e$, by the formula

(3) $$(E(e)f, g)=\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_e(\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}f, g)d\lambda,^{3)} \quad f, g \in \mathfrak{H}.$$

We write $q(x)=a(x)b(x)$, where $a(x)$ is chosen as one of the following two functions:

(4) $$a(x)=|q(x)|^{1/2} \quad \text{or} \quad a(x)=(1+|x|)^{-(3+\delta)/2}.$$

For either chosen $a(x)$, we denote by $A$ and $B$ the multiplicative operators given by $a(x)$ and $b(x)$, respectively. Then $V=AB=BA$. Now we can write

(5) $$(E(e)f, g)=(E_0(e)f, g)-\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_e(AR(\lambda+i\varepsilon)f, B^*R_0(\lambda+i\varepsilon)^*g)d\lambda$$
$$+\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_e(AR(\lambda-i\varepsilon)f, B^*R_0(\lambda-i\varepsilon)^*g)d\lambda,$$

where $E_0(e)$ denotes the resolution of the identity of $L_0$.

Let $Q_0(\kappa)(\operatorname{Im}\kappa\geq 0)$ be the integral operator generated by

(6) $$Q_0(x, y; \kappa)=\frac{a(x)\exp\{i\kappa\,|x-y|\}b(y)}{4\pi\,|x-y|}.$$

It is known that, for either chosen $a(x)$, $Q_0(\kappa)$ is the operator of Hilbert-Schmidt type even for real $\kappa$ (see [5]: § 7). Remark that

(7) $$AR(\zeta)=[I+Q_0(\kappa)]^{-1}AR_0(\zeta), \qquad \kappa^2=\zeta.$$

whenever the bounded inverse $[I+Q_0(\kappa)]^{-1}$ exists.

We can now make use of the Fredholm theory.

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3) $(f, g)$ is used to denote the inner product of $f$ and $g$ in $L^2(E_3)$. The norm of $f$ is denoted by $\|f\|$; i.e., $\|f\|^2=(f, f)$.

**Lemma 1.** *The operator $I+Q_0(\kappa)$ has a bounded inverse if and only if the Schrödinger equation*

$$(8) \qquad [-\Delta+q(x)]\varphi=\kappa^2\varphi$$

*has no non-trivial solutions $\varphi(x,\kappa)$ satisfying the Sommerfeld radiation condition at infinity*:

$$(9) \qquad \varphi(x,\kappa)=O(|x|^{-1}), \quad \lim_{r\to\infty}\int_{|x|=r}\left|\frac{\partial\varphi}{\partial|x|}-i\kappa\varphi\right|^2 d\omega=0.$$

We call a value $\kappa$ for which equation (8) has non-trivial solutions satisfying (9) a singular point of $Q_0(\kappa)$ and denote by $\Sigma$ the set of all singular points of $Q_0(\kappa)$. If $\operatorname{Im}\kappa>0$, then $\kappa\in\Sigma$ if and only if $\mu=k^2$ is a discrete eigenvalue of $L$.

The following lemma will play later an important role.

**Lemma 2.**[4] *$Q_0(\kappa)^2$ vanishes as $|\kappa|\to\infty$; i.e., for given any $\varepsilon>0$ there exists a $\kappa_0=\kappa_0(\varepsilon)>0$ such that*

$$(10) \qquad \|Q_0(\kappa)^2\|<\varepsilon \quad \textit{if} \quad |\kappa|\geq\kappa_0.$$

This proves the following:

**Lemma 3.** *$\Sigma$ forms a bounded closed set in $\operatorname{Im}\kappa\geq 0$. $[I+Q_0(\kappa)]^{-1}$ depends continuously on $\kappa$ except for $\kappa\in\Sigma$ in the sense of the operator norm. Moreover, $\|[I+Q_0(\kappa)]^{-1}\|$ is bounded in the complement in $\operatorname{Im}\kappa\geq 0$ of a neighborhood of $\Sigma$.*

We use also the following lemma due to Kato [3].

**Lemma 4.** *Let $q(x)\in L^{3/2}(E_3)$ and let $a(x)=|q(x)|^{1/2}$. Then*

$$(11) \qquad \int_{-\infty}^{\infty}\{\|AR_0(\lambda+i\varepsilon)f\|^2+\|AR_0(\lambda-i\varepsilon)f\|^2\}d\lambda\leq C_A\|f\|^2,$$

*where $C_A$ is a positive constant independent of $\varepsilon>0$.*

Now let $e=(\alpha,\beta)$ be a (possibly infinite) subinterval of $(0,\infty)$ such that in neighborhoods of $(-\sqrt{\beta},-\sqrt{\alpha})$ and $(\sqrt{\alpha},\sqrt{\beta})$ there exist no singular points of $Q_0(\kappa)$. The existence of such an $e$ is guaranteed by Lemma 3. We return to formula (5). Put $a(x)=|q(x)|^{1/2}$. Then, since $b(x)=a(x)\cdot\{q(x)/|q(x)|\}$, we have $\|B^*R_0(\lambda\pm i\varepsilon)^*g\|\leq\|AR_0(\lambda\mp i\varepsilon)g\|$. Taking (7) and Lemma 3 into account, we have further $\|AR(\lambda\pm i\varepsilon)f\|\leq\text{const}\,\|AR_0(\lambda\pm i\varepsilon)f\|$. Applying Lemma 4, we get[5]

$$\int_e|(AR(\lambda\pm i\varepsilon)f,\,B^*R_0(\lambda\pm i\varepsilon)^*g)|\,d\lambda\leq C_A\|f\|\cdot\|g\|,$$

which proves simultaneously the existence and the boundedness of $E(e)$.

**Theorem 1.**[6] *There exists, for any $e=(\alpha,\beta)$ given as above,*

4) For the proof of this lemma we approximate $q(x)$ by a function in $C^1(E_3)$. A similar estimate for $q(x)\in C^1(E_3)$ is proved in the Lemma of [2].

5) It is clear that $q(x)\in L^{3/2}(E_3)$ if it satisfies condition $(A)$.

6) Cf. J. Schwartz [8]. He obtained results in which $q(x)\in L^1\cap L^\infty$ was assumed together with the existence of an $e=(\alpha,\beta)$.

*a bounded operator* $E(e)$ *satisfying* (3), *which determines the "spectral resolution" of* $L$:

(12) $$E(e)L\subseteq LE(e),$$

(13) $$E(e_1)E(e_2)=E(e_2)E(e_1)=E(e_1\cap e_2).$$

**3. Eigenfunction expansions.** In this section we put $a(x)=(1+|x|)^{-(3+\delta)/2}$. Then, for each $\zeta$ in the resolvent set of $L$, we see from (7) that $AR(\zeta)$ defines an integral operator of Hilbert-Schmidt type. We denote the kernel of $AR(\zeta)$ by $T(x, y; \kappa)$, $\kappa^2=\zeta$. Let $t(x, k; \kappa)=(2\pi)^{-3/2}\int_{E_3} T(x, y; \kappa)e^{-ik\cdot y}dy$, where $k$ denotes a 3-dimensional vector variable. Then we have

(14) $$t(x, k; \kappa)=(|k|^2-\kappa^2)^{-1}\psi(x, k; \kappa), \qquad |k|^2=\sum_{j=1}^{3} k_j^2,$$

where $\psi(\cdot, k; \kappa)\in\mathfrak{H}$ is a solution of the equation

(15) $$\psi(x, k; \kappa)+[Q_0(\kappa)\psi](x, k; \kappa)=(2\pi)^{-3/2}a(x)e^{ik\cdot x}.$$

**Lemma 5.** $\psi(x, k; \kappa)$ *is bounded and continuous in* $E_3\times E_3\times\rho_\Sigma$, *where* $\rho_\Sigma$ *is the complement in* $\operatorname{Im}\kappa\geq 0$ *of a neighborhood of* $\Sigma$. *There exists a positive integer* $n_0$ *such that for any integer* $n\geq n_0$

(16) $$|[Q_0(\kappa)^n\psi](x, k; \kappa)|\leq \text{const}\,(1+|x|)^{-1}a(x)\,\|\psi(\cdot, k; \kappa)\|,$$

*where* $\|\psi\|$ *is bounded in* $k\in E_3$ *and* $\kappa\in\rho_\Sigma$.

It follows in virtue of (2) that $R(\zeta)$ is an integral operator of Carleman type generated by the kernel

(17) $$R(x, y; \kappa)=\frac{\exp\{i\kappa|x-y|\}}{4\pi|x-y|}-\int_{E_3}\frac{\exp\{i\kappa|x-z|\}}{4\pi|x-z|}b(z)T(z, y; \kappa)dz.$$

Letting $r(x, k; \kappa)=(2\pi)^{-3/2}\int_{E_3} R(x, y; \kappa)e^{-ik\cdot y}dy$, we have

(18) $$r(x, k; \kappa)=(2\pi)^{-3/2}(|k|^2-\kappa^2)^{-1}\varphi(x, k; \kappa),$$

where

(19) $$\varphi(x, k; \kappa)=e^{ix\cdot k}-\int_{E_3}\frac{\exp\{i\kappa|x-z|\}}{4\pi|x-z|}b(z)\psi(z, k; \kappa)dz.$$

Put

(20) $$\varphi_\pm(x, k)=\varphi(x, k; \mp|k|), \qquad \pm|k|\notin\Sigma.$$

Then $\varphi_\pm(x, k)$ turns out to be a unique solution of the Lippmann-Schwinger equation

(21) $$\varphi_\pm(x, k)=e^{ix\cdot k}-\int_{E_3}\frac{\exp\{\mp i|k|\cdot|x-z|\}}{4\pi|x-z|}q(z)\varphi_\pm(z, k)dz.$$

A similar function $\varphi_\pm^*(x, k)$ corresponding to $L^*$ is also obtained as a unique solution of (21) with $q(x)$ replaced by $\overline{q(x)}$, if $\pm|k|\notin\Sigma^*$ which is composed of values $-\bar{\kappa}$ corresponding to all $\kappa\in\Sigma$.

Following a way similar to Ikebe's (see [1], § 9), we get

**Lemma 6.** *Let* $e=(\alpha, \beta)$ *be a* (*possibly infinite*) *subinterval of* $(0, \infty)$ *such as given in the previous section and let* $f, g\in C_0^\infty(E_3)$. *Then*

$$(22) \qquad (E(e)f, g)=\int_{\sqrt{\alpha}\le|k|\le\sqrt{\beta}} \hat{f}_{\pm}^{*}(k)\overline{\hat{g}_{\pm}(k)}dk,$$

*where*

$$\hat{f}_{\pm}^{*}(k)=(2\pi)^{-3/2}\int_{E_3}\overline{\varphi_{\pm}^{*}(y, k)}f(y)dy,$$

$$\hat{g}_{\pm}(k)=(2\pi)^{-3/2}\int_{E_3}\overline{\varphi_{\pm}(x, k)}g(x)dx.$$

In the case of a real valued potential, as was proved in [1], relation (22) is extended to $f, g \in \mathfrak{H}$ taking account of $E(e)$ being selfadjoint. In our case however we must directly prove this. Namely

**Lemma 7.** *The mappings $f \to \hat{f}_{\pm}^{*}$ and $g \to \hat{g}_{\pm}$ are bounded maps of $\mathfrak{H}=L^2(E_3)$ into $L^2(K_e)$, where $K_e$ denotes the domain $\{k \in E_3; \sqrt{\alpha}\le|k|\le\sqrt{\beta}\}$.*

*Sketch of proof.* Let us consider the integral

$$X_e^{\pm}(f, g)=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_e (\{R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\}f,\ V^*R(\lambda\mp i\varepsilon)^*g)d\lambda.$$

As we proved the boundedness of $E(e)$, it follows from Lemmas 3 and 4 that this defines a bounded bi-linear form on $L^2(E_3)$. Let $\hat{f}_0(k)$ denote the Fourier image of $f(x)\in L^2(E_3)$. Then the Plancherel theorem shows that

$$X_e^{\pm}(f, g)=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{E_3}\hat{f}_0(k)\overline{\Psi_{\pm}(k, \varepsilon)}dk;$$

$$\Psi_{\pm}(k, \varepsilon)=\int_e \frac{2i\varepsilon}{(\lambda-|k|^2)^2+\varepsilon^2}[V^*R(\lambda\mp i\varepsilon)^*g]\hat{}_0(k)d\lambda.$$

We choose $g(x)$ from $C_0^\infty(E_3)$. Then

$$[V^*R(\lambda\mp i\varepsilon)^*g]\hat{}_0(k)=\int_{E_3}g(x)dx\overline{\int_{E_3}R(x, y: \sqrt{\lambda\mp i\varepsilon})b(y)a(y)e^{ik\cdot y}dy}.$$

Noting the relation[7)]

$$R(\zeta)B=R_0(\zeta)B[I-AR(\zeta)B]\subseteq R_0(\zeta)B[I+Q_0(\sqrt{\zeta})]^{-1}$$

and applying the Lebesgue theorem, we get finally

$$(f, g)-X_e^{\pm}(f, g)=\int_{K_e}\hat{f}_0(k)\overline{\hat{g}_{\pm}(k)}dk,^{8)} \qquad \hat{f}_0(k)\in C_0^\infty(K_e).$$

Since $C_0^\infty(K_e)$ is dense in $L^2(K_e)$, this proves the boundedness of the mapping $g\to\hat{g}_{\pm}$. The boundedness of $f\to\hat{f}_{\pm}^{*}$ is also proved by a similar method. q.e.d.

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7) In fact,

$$I-AR(\zeta)B=I-[I+Q_0(\sqrt{\zeta})]^{-1}AR_0(\zeta)B$$
$$=I-[I+Q_0(\sqrt{\zeta})]^{-1}[I+AR_0(\zeta)B]+[I+Q_0(\sqrt{\zeta})]^{-1}$$
$$\subseteq[I+Q_0(\sqrt{\zeta})]^{-1}.$$

8) If we define a bounded operator $W_{\pm}(e)=E_0(e)-X_{\pm}(e)$, where $(X_{\pm}(e)f, g)=X_e^{\pm}(f, g)$, then we can show that $W_{\pm}(e)$ is the so-called wave operator establishing the similarity between $L_0$ and $L$ (Cf. [6]: §2).

Now we have the following expansion theorem.

**Theorem 2.** *For any $f \in E(e)\mathfrak{H}$, we have*

$$f(x)=(2\pi)^{-3/2}\int_{K_e} \varphi_{\pm}(x, k)\hat{f}_{\pm}^{*}(k)dk; \tag{23}$$

$$\hat{f}_{\pm}^{*}(k)=(2\pi)^{-3/2}\int_{E_3} \overline{\varphi_{\pm}^{*}(y, k)}f(y)dy.$$

**4. The scattering inverse problem.** In virtue of Lemma 3, we see that the distorted plane wave $\varphi_{\pm}(x, k)$ exists for sufficiently large $|k|$. Assume now the additional condition $(A_1)$ on $q(x)$. Then the following assymptotic form of $\varphi_{\pm}$ holds for large $|x|$ (see [2]).

$$\varphi_{\pm}(x, k)=e^{ik\cdot x}+\frac{e^{\mp i|k|\cdot|x|}}{|x|}\theta_{\pm}(n, \nu; |k|)+o(|x|^{-1}), \tag{24}$$

$$\theta_{\pm}(n, \nu; |k|)=-\frac{1}{4\pi}\int_{E_3} \varphi_{\pm}(y, |k|\nu)q(y)e^{\pm i|k|n\cdot y}dy, \tag{25}$$

where $n=x/|x|$ and $\nu=k/|k|$. Let us consider (25) for large $|k|$. Making use of estimate (10), we can obtain that

$$\theta_{\pm}(n, \nu; |k|)=-\frac{1}{4\pi}\int_{E_3} q(y)e^{\pm i|k|(n-\nu)\cdot y}dy+o(1), \tag{26}$$

where by $o(1)$ is meant the term which vanishes as $|k|\to\infty$.

Now for an arbitrary vector $m$ we can choose $|k|$, $n$, and $\nu$ so that $m=|k|(n-\nu)$. We let $|k|\to\infty$ changing $n$ and $\nu$ and preserving the relation $m=|k|(n-\nu)$. Then the limit of the right hand of (26) exists and

$$\lim_{|k|(n-\nu)=m, |k|\to\infty} \theta_{\pm}(n, \nu; |k|)=-\frac{1}{4\pi}\int_{E_3} q(y)e^{\pm im\cdot y}dy. \tag{27}$$

Hence the following theorem holds.

**Theorem 3.** *If a potential $q(x)$ is assumed to satisfy conditions $(A)$ and $(A_1)$, then it is determined uniquely from the assymptotic behavior for $|k|\to\infty$ of the function $\theta_{\pm}(n, \nu; |k|)$.*

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