137. *A Theorem on Paracompactness of Product Spaces*

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1. Introduction. As is well known, the product space of two paracompact Hausdorff spaces is not normal in general. In [4], K. Morita has proved the following:

Let X be a paracompact normal space which is a countable union of locally compact closed subsets, and let Y be a paracompact normal space. Then the product space $X\times Y$ is paracompact and normal.

The purpose of this note is to prove a theorem which is a generalization of Morita's result mentioned above.

Definition. A collection $\{A_\lambda \mid \lambda\in \Lambda\}$ of subsets of a topological space is called *order locally finite*, if we can introduce a total order $<$ in the index set Λ such that for each $\lambda\in\Lambda$ $\{A_\mu \mid \mu<\lambda\}$ is locally finite at each point of A_λ.

Theorem. *If a regular space X has two coverings $\{C_\lambda \mid \lambda\in\Lambda\}$ and $\{U_\lambda \mid \lambda\in\Lambda\}$ such that*

i) C_λ is compact, U_λ is open and $C_\lambda\subset U_\lambda$ for each $\lambda\in\Lambda$, and

ii) $\{U_\lambda \mid \lambda\in\Lambda\}$ is order locally finite,

then for any paracompact regular space Y the product space $X\times Y$ is paracompact.

Let X be a paracompact regular space which is a countable union of locally compact closed subsets. Then there exists a σ-locally finite covering $\{C_\lambda \mid \lambda\in\Lambda\}$ of X such that each C_λ is compact. Moreover, since X is paracompact, there exists a σ-locally finite open covering $\{U_\lambda \mid \lambda\in\Lambda\}$ of X such that U_λ contains C_λ for each $\lambda\in\Lambda$ (see [4]). By Lemma 1 below, a σ-locally finite collection is order locally finite. Therefore our theorem covers certainly Morita's result (in the case when X and Y are regular spaces).

Recently T. Ishii [2] has proved the following:

Let X be the image under a closed continuous mapping of a locally compact and paracompact Hausdorff space, and let Y be a paracompact Hausdorff space. Then the product space $X\times Y$ is a paracompact Hausdorff space.

He has also showed that his result is not covered by Morita's result. As the example below shows, our theorem is not contained in that which is united by Morita's and Ishii's results. We do not, however, know whether our theorem covers Ishii's result.

2. Lemmas. Lemma 1. *Let* $\{A_\lambda \mid \lambda \in \Lambda\}$ *be an order locally finite collection of subsets of a topological space* X, *and let* $\{B_\xi \mid \xi \in \Xi_\lambda\}$ *be a collection of subsets of* A_λ *which is locally finite in* X *for each* $\lambda \in \Lambda$. *Then the collection* $\{B_\xi \mid \xi \in \Xi\}$ *is order locally finite, where* Ξ *is the disjoint union of* Ξ_λ: $\Xi = \cup\{\Xi_\lambda \mid \lambda \in \Lambda\}$.

In particular, a σ*-locally finite collection is order locally finite.*

Proof. By definition, Λ has a total order $<$ such that for each $\lambda \in \Lambda$ $\{A_\mu \mid \mu < \lambda\}$ is locally finite at each point of A_λ. For each $\lambda \in \Lambda$, freely define a total order $<_\lambda$ in Ξ_λ. Next, we define a total order $\prec$ in Ξ as follows. Let $\xi \in \Xi_\lambda$ and $\eta \in \Xi_\mu$. If $\lambda \neq \mu$ and $\lambda < \mu$ then $\xi \prec \eta$, and if $\lambda = \mu$ and $\xi <_\lambda \eta$ then $\xi \prec \eta$.

Now let x be a point of B_ξ, $\xi \in \Xi_\lambda$, then it is a point of A_λ. Hence there exists a neighborhood $U(x)$ of x which intersects only finitely many A_μ for $\mu < \lambda$; let these be $A_{\mu_1}, \cdots, A_{\mu_m}$. Since for each $i = 1, \cdots, m$ the collection $\{B_\xi \mid \xi \in \Xi_{\mu_i}\}$ is locally finite in X, there exists a neighborhood $V_i(x)$ of x which intersects only at most finitely many B_ξ for $\xi \in \Xi_{\mu_i}$. Therefore the neighborhood $U(x) \cap V_1(x) \cap \cdots \cap V_m(x)$ of x intersects only finitely many B_η for $\eta \prec \xi$. This completes the proof.

Lemma 2. *A regular space* X *is paracompact if and only if any open covering of* X *has an order locally finite open refinement.*

Proof. Since the 'only if' part is trivial, we shall prove only the 'if' part. Let $\mathfrak{G}$ be an arbitrary open covering of X, and let $\mathfrak{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ be an order locally finite open refinement of $\mathfrak{G}$. By E. Michael [3, Lemma 1], we need only prove that $\mathfrak{G}$ has a locally finite refinement. By the definition, Λ has a total order $<$ such that for each λ $\{U_\mu \mid \mu < \lambda\}$ is locally finite at each point of U_λ. For each $\lambda \in \Lambda$, define $V_\lambda = U_\lambda - \cup\{U_\mu \mid \mu \lneqq \lambda\}$. To show that the collection $\mathfrak{V} = \{V_\lambda \mid \lambda \in \Lambda\}$ is a locally finite covering of X, let x be a point of X. Then x is contained in some set of $\mathfrak{U}$; let it be U_{λ_0}. By assumption, x is contained only finitely many U_μ for $\mu < \lambda_0$; let these be $U_{\mu_1}, \cdots, U_{\mu_m}$ $(\mu_1 < \cdots < \mu_m)$. Then x is obviously contained in V_{μ_1}, so that $\mathfrak{V}$ is a covering of X. Again by assumption, x has a neighborhood $W(x)$ which intersects only finitely many U_μ for $\mu < \lambda_0$. Then the neighborhood $W(x) \cap U_{\lambda_0}$ of x intersects only finitely many V_ν for $\nu \in \Lambda$, so that $\mathfrak{V}$ is locally finite in X. It is obvious that $\mathfrak{V}$ refines $\mathfrak{G}$. This completes the proof.

3. Proof of Theorem. By Lemma 2, it is sufficient to prove that any open covering $\mathfrak{G}$ of $X \times Y$ has an order locally finite open refinement, for $X \times Y$ is regular. For each element λ of Λ and for each point y of Y, we can find a finite collection $\{H_1, \cdots, H_m\}$ of open subsets of X and an open neighborhood $V(y)$ of y in Y such that

$$H_i \times V(y) \subset \text{some set of } \mathfrak{G} \text{ for } i=1, \cdots, m,$$

and

$$C_\lambda \subset \bigcup_{i=1}^{m} H_i \subset U_\lambda.$$

This is easily verified since C_λ is compact. If we let y range over all the points of Y, the collection of all such $V(y)$ forms an open covering of Y. Since Y is paracompact, this covering has a locally finite open refinement.

Thus for each element λ of Λ we can find a locally finite open covering $\mathfrak{V}_\lambda = \{V_\xi \mid \xi \in \Xi_\lambda\}$ of Y and collections $\mathfrak{H}_\xi$, $\xi \in \Xi_\lambda$, each of which consists of finitely many open subsets of X such that

$$C_\lambda \subset \bigcup\{H \mid H \in \mathfrak{H}_\xi\} \subset U_\lambda \qquad \text{for } \xi \in \Xi_\lambda,$$

and

$$H \times V_\xi \subset \text{some set of } \mathfrak{G} \text{ for } H \in \mathfrak{H}_\xi, \quad \xi \in \Xi_\lambda.$$

By the construction, the collection $\{H \times V_\xi \mid H \in \mathfrak{H}_\xi, \xi \in \Xi_\lambda\}$ of subsets of $U_\lambda \times Y$ is locally finite in $X \times Y$.

On the other hand, the collection $\{U_\lambda \times Y \mid \lambda \in \Lambda\}$ is order locally finite, since the collection $\{U_\lambda \mid \lambda \in \Lambda\}$ is order locally finite. Hence, by Lemma 1, the collection $\{H \times V_\xi \mid H \in \mathfrak{H}_\xi, \xi \in \Xi_\lambda, \lambda \in \Lambda\}$ is order locally finite. This collection is also a covering of $X \times Y$, since the collection $\{C_\lambda \mid \lambda \in \Lambda\}$ is a covering of X. It is obvious that it refines $\mathfrak{G}$. Thus the proof is completed.

4. **Example.** Let X be an uncountable set with a distinguished element x_0. A subset U of X is called open if it does not contain x_0 or if its complement is countable.*) Then X is a regular T_1-space with respect to this topology.

First, we show that X satisfies the condition of our theorem. To the purpose, we define two coverings $\{C_x \mid x \in X\}$ and $\{U_x \mid x \in X\}$ of X which are indexed by the set X as follows. $C_x = \{x\}$ for all x, $U_x = \{x\}$ for $x \neq x_0$ and $U_{x_0} = X$. Then

i) C_x is compact, U_x is open and $C_x \subset U_x$ for each x, and

ii) $\{U_x \mid x \in X\}$ is order locally finite with respect to any total order in X such that x_0 is the first element.

Next, we show that X is not a countable union of locally compact closed subsets. We observe that any compact subset of X is finite. Let A be a locally compact closed subset of X. If A does not contain x_0 then it is countable, because any closed subset not containing x_0 is countable. If A contains x_0, then there exists an open subset V of X such that it contains x_0 and $\overline{V \cap A} \cap A$ is compact.

*) By *countable* we mean countable, finite or empty.

Hence, $X-V$ is countable and $V\cap A$ is finite, so that A is countable. Thus X is not a countable union of locally compact closed subsets, since X is uncountable.

Finally, we show that X is not the image under a closed continuous mapping of a locally compact and paracompact Hausdorff space. To the purpose, deny this assertion. Then X is a k-space by K. Morita [5] or D. E. Cohen [1], since a closed continuous onto mapping is an identification mapping. As we observe above, any compact subset of X is finite. Therefore X is a discrete space, and this is a contradiction.

References

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