131. *The Continuity and the Boundedness of Linear Functionals on Linear Ranked Spaces*

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1. The definition of a bounded set and its properties. Let E be a linear ranked space, by which name we mean a linear space where $\mathfrak{B}_n$ are defined and satisfy axioms (A), (B), (a), (b), (1), (2), (3),[1)]

Definition 1. A subset B in E is called bounded if, for arbitrary n, there is an m, $m \geq n$, and a $V \in \mathfrak{B}_m$ which absorbs B.

Evidently the subset of a bounded set is also bounded. A set consisting of only one point is bounded (cf. axiom (3)).

The linear sum and the union of bounded sets are bounded, too. In fact, let A and B be bounded. For arbitrary n, we can choose an M such that, if $\lambda \geq M$, $\mu \geq M$, then $\phi(\lambda, \mu) \geq n$. Since A and B are bounded, there are $m_1 \geq M$, $m_2 \geq M$, $V_1 \in \mathfrak{B}_{m_1}$, $V_2 \in \mathfrak{B}_{m_2}$ and $\rho_1 > 0$, $\rho_2 > 0$ with $\rho_1 A \subseteq V_1$, $\rho_1 B \subseteq V_2$. Let $\rho = \min.(\rho_1, \rho_2)$. Then

$$\rho(A+B) \subseteq V_1 + V_2, \qquad \rho(A \cup B) \subseteq V_1 \cup V_2 \subseteq V_1 + V_2.$$

Applying (1) for V_1, V_2, E, there exist an $m \geq \phi(m_1, m_2)$, a $V \in \mathfrak{B}_m$ such that $V_1 + V_2 \subseteq V$. Thus we get an $m \geq n$ and a $V \in \mathfrak{B}_m$ which absorbs $A+B$ and $A \cup B$, and therefore they are bounded.

From the properties just proved, it follows that a finite set is bounded.

Proposition 1. *If* $\{\lim x_n\} \neq \phi$, *then the set* $\{x_n\}$ *is bounded* (i.e. *the convergent sequence makes a bounded set*).

Proof. We may assume $\{\lim x_n\} \ni 0$. In fact, $\{\lim x_n\} \ni x$ is equivalent to $\{\lim(x_n - x)\} \ni 0$. If we show that $\{x_n - x\}$ is bounded, we can assert that $\{x_n\} = \{x_n - x\} + \{x\}$, a linear sum of two bounded sets, is bounded. Let $\{\lim x_n\} \ni 0$. Then there exists a sequence $\{V_n\}$ such that

$$V_n \in \mathfrak{B}_{\alpha_n}, \alpha_n \uparrow \infty, V_n \supseteq V_{n+1}, x_n \in V_n (n = 1, 2, \cdots.)$$

For arbitrary given N, we can choose an n_0 such that,

$$\phi(m, \alpha_n) \geq N \text{ for } m \geq n_0, n \geq n_0.$$

Let us denote the set $\{x_n\}$ by A, and let $A = A_1 \cup A_2$, where $A = \{x_n; 1 \leq n \leq n_0 - 1\}$, $A_2 = \{x_n; n \geq n_0\}$. Then, $A_2 \subseteq V_{n_0}$. On the other hand, since A_1 is finite and therefore bounded, there is an $m \geq n_0$, a $V \in \mathfrak{B}_m$, and a $\rho > 0$ with $\rho A_1 \subseteq V$.

Let $\rho' = \min.(\rho, 1)$. Then, $\rho A = \rho'(A_1 \cup A_2) \subseteq V \cup V_{n_0} \subseteq V + V_{n_0}$.

1) M. Washihara: On ranked spaces and linearity. Proc. Japan Acad., **43**, 584-589 (1967).

Applying axiom (1) for V, V_{n_0}, E, we get an $n\geq\phi(m, \alpha_{n_0})$ and a $W\in\mathfrak{V}_n$ such that $V+V_{n_0}\subseteq W$. Thus we have an $n\geq N$ and a $W\in\mathfrak{V}_n$ which absorbs A. Our assertion is proved.

Now, we introduce one new axiom.

$(*_1)$ If both $U\in\mathfrak{V}_m$ and $V\in\mathfrak{V}_n$ absorbs a set B, there exists an $l\geq\max.(m, n)$, and a $W\in\mathfrak{V}_l$ which is included in U, V, and absorbs B.

As is easily seen, if E satisfies $(*)$,[2] $(*_1)$ is also fulfilled.

Proposition 2. *When E satisfies $(*_1)$, for any bounded sequence $\{x_n\}$ and for any sequence $\{\varepsilon_n\}$ with $\varepsilon_n\to 0$, we have $\{\lim \varepsilon_n x_n\}\ni 0$.*

Proof. Let $A=\{x_n\}$. Since A is bounded, there is an $n_1>1$, a $V_1\in\mathfrak{V}_{n1}$ and a $\rho_1>0$ with $\rho_1 A\subseteq V_1$.

Next, we can find an $n_2'>n_1$, a $V_2'\in\mathfrak{V}_{n_2'}$, and a $\rho_2'>0$ with $\rho_2' A\subseteq V_2'$. On account of $(*_1)$, there is an $n_2\geq n_2'$, a $V_2\in\mathfrak{V}_{n_2}$ with $V_2\subseteq V_1\cap V_2'$, and a $\rho_2>0$ such that $\rho_2 A\subseteq V_2$.

Continuing this process, we get sequences $\{n_i\}$, $\{V_i\}$, $\{\rho_i\}$ such that

$$n_i<n_{i+1};\ V_i\in\mathfrak{V}^n{}_i,\ V_i\supseteq V_{i+1};\ \rho_i>0,\ \rho_i A\subseteq V_i.$$

Since $\lim\ \varepsilon_n=0$, we can choose a sequence $\{N_i\}$ such that,

$$N_i<N_{i+1};\ |\varepsilon_n|\leq\rho_i \text{ for } n\geq N_i (i=1, 2, \cdots).$$

Now, let $\alpha_n=n_i$, $U_n=V_i$ when $N_i\leq n<N_{i+1}(i=0, 1, 2, \cdots)$, where $N_0=1$, $n_0=0$, $V_0=E$. Then, $U_n\in\mathfrak{V}_{\alpha_n}$, $U_n\supseteq U_{n+1}$, $\alpha_n\uparrow\infty$.

Moreover, since $|\varepsilon_n|<\rho_i$ for $n\geq N_i$,

$$\varepsilon_n x_n\in\varepsilon_n A=\frac{\varepsilon_n}{\rho_i}(\rho_i A)\subseteq\frac{\varepsilon_n}{\rho_i}V_i\subseteq V_i.$$

Therefore $\varepsilon_n x_n\in U_n$. Thus we have $\{\lim \varepsilon_n x_n\}\ni 0$.

Examples. In preceding paper, we gave three examples of linear ranked spaces; countably normed space Φ, its dual Φ', Schwartz's space $\mathfrak{D}$. Now, let us show that in these spaces boundedness is equivalent to usual one, and the condition $(*_1)$ is valid.

Let B be a bounded set in our sense in the space Φ. Then, for each n, there is an $m\geq n$, and a $\rho>0$ such that $\rho B\subseteq v(m; 0)$. (Note that $\mathfrak{V}_m$ contains only one set $v(m; 0)$.) Hence, for every $\varphi\in B$, $\|\varphi\|_n\leq\|\varphi\|_m<\dfrac{1}{\rho m}$, i.e. $\sup_{\varphi\in B}.\ \|\varphi\|_B<\infty$. Conversely, if for each n, $\sup_{\varphi\in B}.\ \|\varphi\|_n<\infty$, B is bounded in our sense.

Analogously, it is easily verified that a subset B in $\mathfrak{D}$ is bounded if and only if the conditions,

1) there exists some K such that car. $\varphi\subseteq[-K, K]$ for every $\varphi\in B$,

2) for each n, $\sup_{\varphi\in B}.\ \sup_{x}.\ |\varphi^{(n)}(x)|<\infty$ $(n=0, 1, 2, \cdots)$, are ful-

2) M. Washihara: loc. cit.

filled. (Note that, in $\mathfrak{D}$, a neighbourhood of 0 with rank n has the form $v(n, K; 0)=\{\varphi \in \mathfrak{D};\ \text{car.}\ \varphi \subseteq [-K, K],\ \max_{0\leq j\leq n-1}\ \sup_x |\varphi^{(j)}(x)| < \frac{1}{n}\}$).

Finally, a subset B in Φ' is bounded if and only if, for some p, $B\subseteq \Phi'_p$ and $\sup_{f\in B} \|f\|'_p<\infty$. In fact, the boundedness of B implies that for some $n\geq 1$ and for some p, $v(n, p; 0)$ absorbs B. Therefore, there is a $\rho>0$ such that, for every f in B, $\|\rho f\|'_p<\frac{1}{n}$, namely, $\sup_{f\in B} \|f\|'_p\leq \frac{1}{\rho n}$. Conversely, if $\sup_{f\in B} \|f\|'_P<\infty$, B can be absorbed by $v(n, p; 0)$ for any n.

We know that both Φ and $\mathfrak{D}$ satisfy the condition $(*)$, consequently the condition $(*_1)$, too. To prove that $(*_1)$ holds in Φ', let

$$U=v(m, p; 0),\ V=v(n, q; 0),\ \rho_1>0,\ \rho_2>0,\ \rho_1 B\subseteq U,\ \rho_2 B\subseteq V.$$

We may assume $n\geq m$. Now, if $p\geq q$, then $V\subseteq U$, and therefore we can take V itself as W. On the other hand, if $p<q$, letting $W=v(n, p; 0), \rho=\frac{m\rho_1}{n}$, we have $W\in \mathfrak{B}_n$, $W\subseteq U\cap V$. In addition, for $f\in B$, since $\rho_1 f\in U$, $\|\rho_1 f\|'_p<\frac{1}{m}$, consequently $\|\rho f\|'_p<\frac{\rho}{m\rho_1}=\frac{1}{n}$, i.e. $\rho f\in W$. Hence, $\rho B\subseteq W$. Thus our assertion is proved.

2. The continuity and the boundedness of linear functionals.

Definition 2. A linear functional f on a linear ranked space E is called continuous if $\{\lim x_n\}\ni 0$ implies $\lim f(x_n)=0$. f is called bounded if for any bounded set B in E, $\sup_{x\in B} |f(x)|<\infty$.

Proposition 3. *Let E satisfy the condition $(*_1)$. If a linear functional f on E is continuous, f is bounded.*

Proof. Suppose that f is not bounded. There exists a bounded set B such that $\sup_{x\in B} |f(x)|=\infty$. We can find a sequence $\{x_n\}$ in B with $|f(x_n)|>n\ (n=1, 2, \cdots)$. From Proposition 2, $\left\{\lim \frac{1}{n}x_n\right\}\ni 0$, while $f\left(\frac{1}{n}x_n\right)\not\to 0$. Hence f is not continuous.

The converse of this proposition is valid if E satisfies following condition:

(4) There exists a non-negative function $\chi(\lambda, \mu)$ defined for $\lambda\geq 0, \mu\geq 1$, and the following holds; if $U\in \mathfrak{B}_m$, $V\in \mathfrak{B}_n$, $U\subseteq V$, and $m\geq \chi(n, k)$, then $kU\subseteq V$. To prove this, we need following lemma.

Lemma. *Let E satisfy* (4). *If* $\{\lim x_n\}\ni 0$, *there exists a sequence of positive numbers* $\{M_n\}$ *such that* $M_n\uparrow\infty$, *and* $\{\lim M_n x_n\}\ni 0$.

Proof. From hypothesis there is a sequence $\{V_n\}$ such that

$$V_n\in \mathfrak{B}_{\alpha_n},\ V_n\supseteq V_{n+1},\ \alpha_n\uparrow\infty,\ x_n\in V_n.$$

First, we choose an $n_1>1$, such that $\alpha_{n_1}\geq \chi(\alpha_1, 2)$, $\alpha_1<\psi(\alpha_{n_1}, 2)$.

(This is possible because $\lim_{n\to\infty} \psi(\alpha_n, 2)=\infty$.) Since $V_{n_1} \in \mathfrak{V}^{\alpha}{}_{n_1}$, $V_1 \in \mathfrak{V}_{\alpha_1}$, $V_{n_1} \subseteq V_1$, from (4), we have $2V_n \subseteq V_1$.

Applying axiom (2) for $U=V_{n_1}$, $V=V_1$, $\alpha=2$, there is a $\beta_1 \geq \psi(\alpha_{n_1}, 2)$ (consequently, $\beta_1 > \alpha_1$), and a $W_1 \in \mathfrak{V}_{\beta_1}$ with $2V_{n_1} \subseteq W_1 \subseteq V_1$.

Next, we choose an $n_2 > n_1$, such that $\alpha_{n_2} \geq \chi(\alpha_{n_1}, 2)$, $\beta_1 < \psi(\alpha_{n_2}, 4)$. From (4), $2V_{n_2} \subseteq V_{n_1}$, and therefore $4V_{n_2} \subseteq 2V_{n_1} \subseteq W_1$. Applying again axiom (2) for $U=V_{n_2}$, $V=W_1$, $\alpha=4$, there is a $\beta_2 \geq \psi(\alpha_{n_2}, 4)$ (consequently, $\beta_2 > \beta_1$), and a $W_2 \in \mathfrak{V}_{\beta_2}$, with $4V_{n_2} \subseteq W_2 \subseteq W_1$.

Continuing this process, we get sequences $\{n_i\}$, $\{\beta_i\}$, $\{W_i\}$ such that

$$n_i < n_{i+1},\ \beta_i < \beta_{i+1};\ W_i \in \mathfrak{V}_{\beta i},\ W_i \supseteq W_{i+1};\ 2^i V_{n_i} \subseteq W_i (i=0, 1, 2, \cdots)$$

where $n_0=1$, $\beta_0=\alpha_1$, $W_0=V_1$.

Let $\gamma_n=\beta_i$, $U_n=W_i$, $M_n=2^i$ when $n_i \leq n < n_{i+1} (i=0, 1, 2, \cdots)$ Then,

$$U_n \in \mathfrak{V}_{\gamma_n},\ U_n \supseteq U_{n+1},\ \gamma_n \uparrow \infty,\ M_n x_n \in U_n (n=1, 2, \cdots)$$

Hence $\{\lim M_n x_n\} \ni 0$, while $M_n \uparrow \infty$. Thus our assertion is proved.

Now, suppose that f is not continuous. Then there exists a sequence $\{x_n\}$ such that $\{\lim x_n\} \ni 0$, $f(x_n) \not\to 0$. Without loss of generality we can suppose that $|f(x_n)| \geq 1$. From the lemma just proved, there is a sequence $\{M_n\}$ such that $M_n \uparrow \infty$, $\{\lim M_n x_n\} \ni 0$. From Proposition 1, $\{M_n x_n\}$ is bounded, while, $|f(M_n x_n)| \geq M_n$ and therefore $\sup_n |f(M_n x_n)| = \infty$. Hence f is not bounded.

Thus, following proposition is proved.

Proposition 4. *Let E satisfy* (4). *If a linear functional f on E is bounded, f is continuous.*

We know that $(*_1)$ holds in Φ, $\mathfrak{D}$, Φ'. Let us prove that in these spaces (4) also holds. In any case, we may take $\chi(\lambda, \mu)=\lambda\mu$.

First, let $U=v(m; 0)$, $V=v(n; 0)$ and $U \subseteq V$, $m \geq n$k. Then for each $\varphi \in U$

$$\| K\varphi \|_n \leq \| K\varphi \|_m < \frac{K}{m} \leq \frac{1}{n}$$

therefore $k\varphi \in V$. Hence $kU \subseteq V$. Thus (4) holds in Φ.

Next, let $U=v(m, K; 0)$, $V=v(n, L; 0)$, and $U \subseteq V$, $m \geq nk$. Then, necessarily, $K \leq L$. It is easily verified that $kU \subseteq V$. Thus $\mathfrak{D}$ satisfies (4), too.

Finally, let $U=v(m, p; 0)$, $V=v(n, q; 0)$, and $U \subseteq V$, $m \geq nk$. Since $U \subseteq V$, $p \leq q$. If $f \in U$, then $\| kf \|_q' \leq \| kf \|_p' < \frac{K}{m} \leq \frac{1}{n}$, consequently, $kf \in V$. Hence $kU \subseteq V$. Thus Φ' also satisfies (4).