163. On Extension of Almost Periodic Functions

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In this note, we shall prove an extension theorem of almost periodic functions on a topological semifield. For the concept of topological semifield, see [1] and [2].

Let E_1 be an arbitrary topological semifield, E_2 a complete topological semifield. We consider the set M of all bounded function $f: E_1 \to E_2$. For $f, g \in M$, we define its distance by

$$\rho(f, g) = \sup_{x \in E_1} d(f(x), g(x)) = \sup_{x \in E_1} |f(x) - g(x)|$$

where $|x|$ denotes the absolute value of x. As easily seen, $\rho(f, g)$ satisfies the well known axioms on a metric. Then M is a metric space over a topological semifield E_2. E_2 is complete, so M is complete.

Definition 1. A function $f(x)(x \in E_1)$ is called *almost periodic*, if it is continuous on E_1, and if for every neighborhood $U^q_{0,\varepsilon}$ (in E_2) there exists a neighborhood $U^q_{a,a+\delta}$ (in E_1) containing at least one element $y = y(\varepsilon)$ for which the relation $d(f(x+y), f(x)) \in U^q_{0,\dot{\varepsilon}}$ for all $x \leftarrow U^q_{a,a+\delta}$ holds.[1] Such an element $y(\varepsilon)$ is called an ε-period of the function f.

Then every almost periodic function is bounded on the topological semifield and therefore belongs to the space M.

Definition 2. *A* set K of a metric space X over a topological semifield is called *ε-net* for the set M of the space, if for every element $f \in M$ there exists an element $f_\varepsilon \in K$ such that $\rho(f, f_\varepsilon) \in U^q_{0\,\varepsilon}$.

Proposition (Extension of Hausdorff's theorem). In order that a set M in a metric space X over a topological semifield be compact, it is necessary that for every $\varepsilon > 0$ there exists a finite ε-net for M. If the space X is complete, then the condition is also sufficient.

Proof of necessity: We assume that M is compact. Let f_1 be an arbitrary element of M. If $\rho(f, f_1) \in U^q_{0,\varepsilon}$ for all $f \in M$, then a finite ε-net exists. If, however, this is not the case, then there exists an element $f_2 \in M$ such that $\rho(f_1, f_2) \notin U^q_{0,\dot{\varepsilon}}$. If for every element $f \in M$ either $\rho(f_1, f) \in U^q_{0,\varepsilon}$ or $\rho(f_2, f) \in U^q_{0,\varepsilon}$, then we have found a finite ε-net. If, however, this does not hold, then there exists an element f_3 such that

$$(f_1, f_3) \notin U^q_{0,\dot{\varepsilon}}, \qquad (f_2, f_3) \notin U^q_{0,\dot{\varepsilon}}.$$

Continuing this way, we obtain elements $f_1, f_2, \cdots, f_n$ for which $\rho(f_i, f_j) \notin U^q_{0,\dot{\varepsilon}}$ if $i \neq j$. There exist two possibilities. Either the

1) We put $U^q_{0,\varepsilon} = \{x \in E_1 \mid 0 < x(q) > \varepsilon\}$, $U^q_{0,\dot{\varepsilon}} = \{x \in E_1 \mid 0 < x(q) \leq \varepsilon\}$.

procedure ceases after the kth step, i.e., for every $f \in M$ one of the relation

$$\rho(f_i, f) \in U^q_{0,\varepsilon}, \qquad i=1, 2, \cdots, k,$$

holds and the $f_1, f_2, \cdots, f_k$ form a finite ε-net for M, or we can continue indefinitely the present procedure. The latter, however, cannot occur, since otherwise we would obtain an infinite sequence $\{f_n\}$ of elements such that $\rho(f_i, f_j) \in U^q_{0,\bar{\varepsilon}}$ for $i \neq j$, and neither this sequence nor any of its subsequences would converge. This is a contradiction to the hypothesis that M is compact.

Proof of sufficiency: We assume that the space X is complete and that to every $\varepsilon > 0$ there is a finite ε-net for M. We choose a null sequence $\{\varepsilon_n\}$. For every ε_n we construct a finite ε_n-net $[f_1^{(n)}, f_2^{(n)}, \cdots, f_{k_n}^{(n)}]$ for the set M. Then we choose an arbitrary infinite subset $S \subset M$. Around every element $f_1^{(1)}, f_2^{(1)}, \cdots, f_{k_1}^{(1)}$ of the ε_1-net we place a closed sphere B_{ε_1} such that $\rho(f, g) \in U^q_{0,2\bar{\varepsilon}_1}$ for every f, $g \in B_{\varepsilon_1}$. Then every elements of S is contained in one of these spheres. Since the number of the sphere is finite, there is at least one sphere containing an infinite set of elements of S. We denote this subset of S by S_1. Around every element $f_1^{(2)}, f_2^{(2)}, \cdots, f_{k_2}^{(2)}$ of the ε_2-net we place a closed sphere B_{ε_2} such that $\rho(f, g) \in U^q_{0\,2\bar{\varepsilon}_2}$ for every $f, g \in B_{\varepsilon_2}$. By the same reasoning as above, we obtain an infinite set $S_2 \subset S_1$, situated in one of the constructed spheres B_{ε_2}. Continuing this procedure, we obtain a sequence of infinite subsets of S: $S_1 \supset S_2 \supset \cdots \supset S_n, \cdots$, where the subset S_n is contained in a closed sphere B_{ε_n}.

Now we choose an element $f_1 \in S_1$, an element $f_2 \in S_2$, different from f_1 an element $f_3 \in S_3$, different from f_1 and f_2, and so on, and we obtain a sequence of elements $S_\omega = \{f_1, f_2, \cdots, f_n, \cdots\}$ which is a Cauchy-sequence, $f_n \in S_n$ and $f_{n+p} \in S_{n+p}$ for every natural number p implies

$$\rho(f_{n+p}, f_n) \in U_{0,2\varepsilon_n} \to 0 \quad \text{as} \quad n \to \infty.$$

By hypothesis the space X is complete, so the sequence S_ω converges to an element $f \in X$. This proves the compactness of the set M.

Corollary. *A set M of a complete metric space X over a topological semifield is compact if and only if for every $\varepsilon > 0$ there exists a compact ε-net for M.*

Proof. Let K be a compact $\varepsilon/2$-net for the set M. Applying the Proposition to K, we find that there exists a finite $\varepsilon/2$-net K_0 for K. Then K_0 is a finite ε-net for M. For every element $f \in M$ there exists an element $f_1 \in K$ such that $\rho(f, f_1) \in U^q_{0,\varepsilon/2}$. Furthermore, for every element $f_1 \in K$ there exists an element $f_2 \in K_0$ such that $\rho(f_1, f_2) \in U^q_{0,\varepsilon/2}$. Consequently, for every element $f \in M$, there exists a element f_2, such that

$$\rho(f_1, f_2) \ll \rho(f, f_1) + \rho(f_1, f_2) \in U^q_{0,\varepsilon/2} + U^q_{0,\varepsilon/2} \subseteq U^q_{0,\varepsilon},$$

i.e., K_0 is a finite ε-net for M. Since the space X is complete, we conclude by proposition that M is compact.

Then we shall prove the following

Theorem. *A set P of almost periodic functions is compact in the sense of the metric of M if and only if*

(1) *the functions of the set P are uniformly bounded and equicontinuous.*

(2) *for every neighborhood $U^q_{0,\eta}$ (in E_2), there exists a neighborhood $U^q_{a,a+l}$ (in E_1) containing an element h which is an η-period for all functions of the set P.*

Proof of necessity: The proof of (1) is analogous to the proof of the corresponding assertion in the generalization of Ascoli-Arzela's theorem on the topological semifield [4]. We consider condition (2).

Since P is compact, for every $\eta > 0$, there exists a finite $\eta/3$-net for the set P. Let us denote these elements by $f_1, f_2, \cdots, f_n$. Then, for every element $f \in P$ there exists an element $f_i (1 \leq i \leq n)$ such that $\rho(f, f_i) \in U^q_{0,\eta/\varepsilon}$. There exists a number $l > 0$ such that every neighborhood $U^q_{a,\alpha+l}$ containing an element h which is an $\eta/3$-period for all f_i, $i = 1, 2, \cdots, n$:

$$d(f_i(x+h), f_i(x)) \in U^q_{0,\eta/3} \quad \text{for all} \quad x \in E_1 (i = 1, 2, \cdots, n)$$

(The proof is analogous to one of the corresponding assertion on the real number line $-\infty < x < +\infty$ which has shown by Bohr).

Since, on the other hand, $\{f_i\}$ is an $\eta/3$-net for P, there exists for every function $f \in P$ an f_i such that

$$d(f(x+h), f(x)) \ll d(f(x+h), f_i(x+h)) + d(f_i(x+h), f_i(x)) + d(f_i(x), f(x)) \in U^q_{0,\eta/3} + U^q_{0,\eta/3} U^q_{0,\eta/3} \subset U^q_{0,\eta}$$

for $x \in E_i$. Therefore h is an η-period for all $f \in P$ and we complete the necessity of (2).

Proof of sufficiency: We assume that for a set P of almost periodic function, (1) and (2) are fulfilled and choose a neighborhood $U^q_{0,\eta}$ (in E_2). Let $l = l(\eta)$ be determined such that every neighborhood $U^q_{a,a+l}$ has an η-period for all $f \in P$. We associate with every $f \in P$ a function $\bar{f}$ defined by

$$\bar{f}(x) = \begin{cases} f(x), & \text{if} \quad x \in U^q_{-l,l}, \\ f(x - r_n), & \text{if} \quad \begin{cases} x \in U_{nl,(n+1)l} (n = 1, 2, 3, \cdots,), \\ x \in U_{nl,(n+1)l} (n = -2, -3, \cdots,), \end{cases} \end{cases}$$

where r_n is an η-period for all $f \in P$, and its period lies in the neighborhood $U^q_{nl,(n+1)l}$. We denote the set of all $\bar{f}$ by P_η. By condition (1), all functions $\bar{f} \in P_\eta$ satisfy the conditions of the theorem of Ascoli-Arzela (in the sense of extension) on the neighborhood

$U^q_{-i,i}$. Therfore P_η is compact in the sence of uniform convergence on the neighborhood $U^q_{-i,i}$. By $x-r_n \in U^q_{-i,i}$, a sequence of functions $\bar{f}$ which converges uniformly on the neighborhood $U^q_{-i,i}$ converges uniformly also on the entire topological semifield E_1 by definition of these functions. Consequently the set P_η is compact in the sense of uniform convergence on the entire topological semifield E_1, i.e., in the sense of the metric of the space M. For arbitrary $f \in P$ and the corresponding $\bar{f} \in P_\eta$,

$$d(f(x), \bar{f}(x))=0 \qquad \text{if} \quad x \in U^q_{-i,i}$$

and

$$d(f(x), \bar{f}(x))=d(f(x), f(x-r_n)), \quad \text{if} \quad \begin{cases} x \in U^q_{nl,(n+1)l}(n=1, 2, \cdots,), \\ x \in U^q_{nl,(n+1)l}(n=-2, -3, \cdots,). \end{cases}$$

Since r_n is an η-period for f, for an arbitrary x we have

$$d(f(x), \bar{f}(x)) \in U^q_{0,\eta}.$$

Hence the compact set P_η forms an η-net for P in the space M. By corollarty to Proposition, P is compact and therefore, we have shown that the conditions (1) and (2) are sufficient.

References

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