192. A Note on M-Spaces

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Let X and Y be topological spaces and f be a closed, continuous mapping from X onto Y such that for each $y \in Y$, $f^{-1}(y)$ is countably compact. Such an f is called a quasi-perfect mapping. Moreover, if $f^{-1}(y)$ is compact, f is called a perfect mapping. In [5], K. Morita defined P-spaces and M-spaces. According to [5] an M-space is a P-space, and in order that X be an M-space it is necessary and sufficient that there exist a metric space Y and a quasi-perfect mapping from X onto Y. We study the M-spaces with some compactness or completeness by use of the above mappings. Let X be a completely regular T_1 space which admits a complete uniformity. Then X is not necessarily paracompact by H. H. Corson [2]. In this note, it will be shown that X is paracompact if the space X concerned is an M-space. The author wishes to thank Prof. K. Nagami who has given useful advices. In this paper, for topological spaces no separation axiom is assumed unless otherwise provided.

Theorem 1. *Every M-space is countably paracompact.*

Theorem 2. *If X is a pseudo-compact M-space, then X is countably compact. Therefore in a pseudo-compact space, to be an M-space is equivalent to countable compactness.*

As will be seen later, a P-space is not countably paracompact in general and a pseudo-compact P-space need not be countably compact.

Proof of Theorem 1. The result follows from the above characteristic property of M-spaces and the following lemma.

Lemma. *If f is a quasi-perfect mapping from X onto Y and Y is countably paracompact (or countably compact), then X is countably paracompact (or countably compact, respectively).*

This lemma is known in [4].

Proof of Theorem 2. Let f be a quasi-perfect mapping from X onto a metric space Y. Since Y is pseudo-compact metric, Y is compact. Hence X is countably compact by the above lemma.

It is known in [5] every countably compact space is an M-space and every normal $P(1)$-space is countably paracompact. It is to be noted that *a pseudo-compact, locally compact Hausdorff P-space is not necessarily countably paracompact and hence not an M-space.* The Tychonoff plank $X=[0, \omega_1]\times[0, \omega]-t$, where t is the point

(ω_1, ω), is such a space. X is not countably paracompact, since X is not countably compact but pseudo-compact, completely regular. We shall see X is a P-space. For a set Ω of any power and for any family $\{G(\alpha_1, \cdots, \alpha_i): \alpha_1, \cdots, \alpha_i \in \Omega; i=1, 2, \cdots\}$ of open subsets of X such that
$G(\alpha_1, \cdots, \alpha_i) \subset G(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$ for $\alpha_1, \cdots, \alpha_i, \alpha_{i+1} \in \Omega; i=1, 2, \cdots,$
we show the existence of a family

$$\{F(\alpha_1, \cdots, \alpha_i): \alpha_1, \cdots, \alpha_i \in \Omega; i=1, 2, \cdots\}$$

of closed subsets of X satisfying the two conditions below:

i) $F(\alpha_1, \cdots, \alpha_i) \subset G(\alpha_1, \cdots, \alpha_i)$ for $\alpha_1, \cdots, \alpha_i \in \Omega; i=1, 2, \cdots,$

ii) $X = \bigcup_{i=1}^{\infty} F(\alpha_1, \cdots, \alpha_i)$ for any sequence $\{\alpha_i\}$ such that $X = \bigcup_{i=1}^{\infty} G(\alpha_1, \cdots, \alpha_i)$.

Let $A = [0, \omega_1) \times \{\omega\}$ and $A_n = [0, \omega_1] \times [0, n]$ for each integer $n \geq 0$. Then A is countably compact and A_n is compact. For each $G(\alpha_1, \cdots, \alpha_i)$,
let

$$E_1(\alpha_1, \cdots, \alpha_i) = \begin{cases} \phi & \text{if } A \not\subset G(\alpha_1, \cdots, \alpha_i) \\ A & \text{if } A \subset G(\alpha_1, \cdots, \alpha_i), \end{cases}$$

$$E_2(\alpha_1, \cdots, \alpha_i) = \begin{cases} \phi & \text{if } A_0 \not\subset G(\alpha_1, \cdots, \alpha_i) \\ A_n & \text{if } A_n \subset G(\alpha_1, \cdots, \alpha_i),\ A_{n+1} \not\subset G(\alpha_1, \cdots, \alpha_i) \\ & \quad \text{for some } n, \end{cases}$$

and

$$F(\alpha_1, \cdots, \alpha_i) = E_1(\alpha_1, \cdots, \alpha_i) \cup E_2(\alpha_1, \cdots, \alpha_i).$$

Then each $F(\alpha_1, \cdots, \alpha_i)$ is closed and

$$\{F(\alpha_1, \cdots, \alpha_i): \alpha_1, \cdots, \alpha_i \in \Omega; i=1, 2, \cdots\}$$

is a family satisfying the above conditions.

A space X is said to be *point-paracompact* iff every open covering of X has a point-finite open covering as a refinement.

Theorem 3. *Let f be a perfect mapping from X onto Y and X, Y be completely regular T_1 spaces. If Y admits a complete uniformity (in the sense of Tukey), then X does also.*

Proof. Let $\{\mathfrak{V}_\mu\}$ be a complete uniformity of Y and $\{\mathfrak{U}_\lambda\}$ be the uniformity of all open normal coverings of X. Let $\mathfrak{F}$ be any family of X with the finite intersection property such that for every λ, there exists an element A of $\mathfrak{F}$ satisfying $A \subset \mathrm{St}(x, \mathfrak{U}_\lambda)$ for some x. Without loss of generality it is supposed that $\mathfrak{F}$ is multiplicative. Now $f(\mathfrak{F})$ has the finite intersection property. Since $f^{-1}(\mathfrak{V}_\mu)$ is an open normal covering of X for each μ, there exists a λ with $\mathfrak{U}_\lambda = f^{-1}(\mathfrak{V}_\mu)$. By the definition of $\mathfrak{F}$, there exists an A of $\mathfrak{F}$ such that $A \subset \mathrm{St}(x, \mathfrak{U}_\lambda)$ for some x, Therefore

$$f(A) \subset \mathrm{St}(f(x), \mathfrak{V}_\mu).$$

Since $\{\mathfrak{V}_\mu\}$ is complete, $\bigcap_{A\in\mathfrak{F}}\overline{f(\bar{A})}\neq\phi$. Let $y_0\in\bigcap_{A\in\mathfrak{F}}\overline{f(\bar{A})}$. Then for every $A\in\mathfrak{F}$, $f^{-1}(y_0)\cap\bar{A}\neq\phi$, by the closedness of f. Since the collection $\{f^{-1}(y_0)\cap\bar{A}: A\in\mathfrak{F}\}$ has the finite intersection property by the multiplicative condition of $\mathfrak{F}$ and $f^{-1}(y_0)$ is compact, $f^{-1}(y_0)\cap\bigcap_{A\in\mathfrak{F}}\bar{A}\neq\phi$. Therefore $\bigcap_{A\in\mathfrak{F}}\bar{A}\neq\phi$.

Theorem 4. *Let X be a completely regular T_1 M-space. Then the following are equivalent*:

i) *X admits a complete uniformity.*

ii) *X is paracompact.*

iii) *X is point-paracompact.*

Proof. It is obvious that ii) implies iii). Let X satisfy the condition i), and let f be a quasi-perfect mapping from X onto a metric space Y. Then for each $y\in Y$, $f^{-1}(y)$ is compact by the completeness of X. Hence f is perfect. Therefore X is paracompact since Y is paracompact. Thus i) implies ii). Let X be a point-paracompact M-space and f be a quasi-perfect mapping from X onto a metric space Y. Since $f^{-1}(y)$ is closed for each $y\in Y$, $f^{-}(y)$ is point-paracompact. Then $f^{-1}(y)$ is compact by R. Arens and J. Dugundji [1]. Hence f is perfect. Since every metric space has a complete uniformity, X has a complete uniformity by Theorem 3. Thus iii) implies i).

Theorem 5. *Let X be a completely regular T_1 M-space which admits a unique uniformity. Then X is countably compact.*

This is proved by the fact that a metric space which admits a unique uniformity is compact.

By Theorems 4 and 5 we obtain the following

Corollary. *Let X be a completely regular T_1 M-space. Then the following are equivalent*:

i) *X admits a unique, complete uniformity.*

ii) *X is compact.*

iii) *X is pseudo-compact and point-paracompact.*

References

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