7. On Connections of Geometric Structures

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Let G_0 and $\tilde{G}$ be Lie groups and $\rho: \tilde{G} \to G_0$ be a homomorphism. G_0 acts on another Lie group K from the left distributively:

$$a \cdot (k_1 \cdot k_2) = (a \cdot k_1) \cdot (a \cdot k_2) \text{ for } a \in G_0 \text{ and } k_1, k_2 \in K.$$

Let $\Theta: \tilde{G} \to K$ be a C^∞-mapping such that

$$(1) \qquad \Theta(a \cdot b) = \{\rho(b^{-1}) \cdot \Theta(a)\} \cdot \Theta(b).$$

Then clearly $G = \{a \in G: \Theta(a) = 1\}$ is a closed subgroup of $\tilde{G}$ and we have the

Proposition 1. *There is a canonical action of $\tilde{G}$ on K from the right, defined by*

$$(2) \qquad k \cdot a = \{\rho(a^{-1}) \cdot k\} \Theta(a), \text{ where } k \in K \text{ and } a \in \tilde{G}.$$

Assume that $P(M, \tilde{G})$ be a C^∞-differentiable principal fibre bundle over a C^∞-manifold M of n dimensions. The we have two induced fibre bundles $T(M, K, \tilde{G})$ and $B(M, K, \tilde{G})$ over M with fibre K, associated with $P(M, \tilde{G})$, determined by ρ and the action of $\tilde{G}$ on K in Proposition 1, respectively. A C^∞-cross-section of $T(M, K, \tilde{G})$ is called, by the abuse of language, as a tensor field on M of type ρ, while we define a *connection of type* $(P(M, \tilde{G}), \rho, \Theta)$ as a C^∞-cross-section ω of $B(M, K, \tilde{G})$.

Proposition 2. *If ω_1 and ω_2 are two connections of type $(P(M, \tilde{G}), \rho, \Theta)$, then $\omega_1 \cdot \omega_2^{-1}$ is a tensor field on M of type ρ.*

It must be remarked that in the above proposition $\omega_1^{-1} \cdot \omega_2$ is not necessarily a tensor field of type ρ, unless K is abelian.

Our definition generalizes that of Gunning [2], who studied the case where K is a vector space, $G_0 = GL(K)$ and $P(M, \tilde{G})$ is $F^r(M)$ we define below.

Proposition 3. *The definition of the connection above includes those of principal fibre bundles of Ehresmann* [1], *of vector bundles as splittings of short exact sequences* (cf. P. Libermann [6]), *and of the bundles of higher order defined by Ehresmann* (cf. N. V. Quê [7]), *provided that $P(M, \tilde{G})$, ρ, Θ are suitably chosen.*

The proof is easily checked in all cases.

In the applications important is the following

Proposition 4. *If M is paracompact and K is a connected nilpotent Lie group, then there is a connection of type $(P(M, \tilde{G})\, \rho, \Theta)$.*

In the following we consider affine connections of higher order, as an example. Let $F^r(M)$ (resp. $F^{(r)}(M)$) be the set of all invertible holonomic (resp. semi-holonomic) r-jets with source $O \in R^n$ and targets in M. Then $F^r(M)$ (resp. $F^{(r)}(M)$) is a differentiable principal bundle over M with fibre $G^r(n)$ (resp. $G^{(r)}(n)$). Then we have surjective homomorphisms: $\pi^r: G^r(n) \to GL(n, R)$ (resp. $\pi^{(r)}: G^{(r)}(n) \to GL(n, R)$) and $s^r: G^{(r)}(n) \to G^r(n)$, such that $\pi^{(r)} = \pi^r \cdot s^r$, where s^r is called the symmetrization operator. We put $K^r = \operatorname{Ker} \pi^r$ (resp. $K^{(r)} = \operatorname{Ker} \pi^{(r)}$). On the other hand, we have canonical injections of Lie groups:

$i^r: GL(n, R) \to G^r(n)$ (resp. $i^{(r)}: GL(n, R) \to G^{(r)}(n)$) and $j^r: G^r(n) \to G^{(r)}(n)$, such that $j^r \cdot i^r = i^{(r)}$. Then we may consider $GL(n, R) \subset G^r(n) \subset G^{(r)}(n)$, by which we define the mapping $\Theta^r: G^r(n) \to K^r$ (resp. $\Theta^{(r)}: G^{(r)}(n) \to K^{(r)}$), satisfying the condition (1). Namely, $\Theta^{(r)}(a) = \rho(a^{-1}) \cdot a$, $\Theta^r = \Theta^{(r)} \mid G^r(n)$ and the action of $GL(n, R)$ on $K^{(r)}$ is defined by $a \circ k = a \cdot k \cdot a^{-1}$ with respect to the multiplication in $G^{(r)}(n)$, which is clearly associative. A connection ω^r(resp. $\omega^{(r)}$) of type $(F^r(M), \pi^r, \Theta^r)$ (resp. $(F^{(r)}(M), \pi^{(r)}, \Theta^{(r)})$) is called a *symmetric affine connection* (resp. *affine connection*) *of order* r. If $r=2$, it is the same as the classical one. From Proposition 4, follows the

Proposition 5. *On a paracompact differentiable manifold M, there always exists a symmetric affine connection (resp. affine connection with torsion) of order r.*

Here, the torsion tensor field T^r of an affine connection $\omega^{(r)}$ of order r is defined by $T^r = \omega^{(r)} \cdot s^r(\omega^{(r)})^{-1}$. ω^r(resp. $\omega^{(r)}$) is equivalent to a group reduction of the group $G^r(n)$ (resp. $G^{(r)}(n)$) of $F^r(M)$ (resp. $F^{(r)}(M)$) to $GL(n, R)$.

Proposition 6. *The vanishing of the torsion tensor field of an affine connection $\omega^{(r)}$ of order r is the same as the obstruction condition that the group reduction of $F^{(r)}(M)$ to $GL(n, R)$, determined by $\omega^{(r)}$is factored through by the canonical group reduction $j^r: F^r(M) \to F^{(r)}(M)$.*

If there is given a closed Lie subgroup of $G^r(n)$ (resp. $G^{(r)}(n)$), for instance, as in the case of projective structure without torsion (resp. with torsion) (cf. Gunning [2], and Kobayashi and Nagano [3]), we can discuss the connection theory for the G-structure of r-th order.

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