4. Relations between Unitary $\rho$-Dilatations and Two Norms

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Introduction. In this paper we discuss classes of power bounded operators on a Hilbert space $H$ and we use the notations and terminologies of [5]. Following [1] [2] [5], an operator $T$ on $H$ possesses a unitary $\rho$-dilatation if there exists a Hilbert space $K$ containing $H$ as a subspace, a positive constant $\rho$ and a unitary operator $U$ on $K$ satisfying the following representation

\[ T^n = \rho^* P U^n \quad (n=1, 2, \cdots) \]

where $P$ is the orthogonal projection of $K$ on $H$. Put $C_\rho$ the class of operators, whose powers $T^n$ admit a representation (1).

It is well known that $T \in C_1$ is characterized by $\| T \| \leq 1$. Moreover $T \in C_\alpha$ is characterized by $\| T \|_\alpha \leq 1$, where $\| T \|_\alpha$, usually called the numerical radius of $T$, is defined by

\[ \| T \|_\alpha = \sup \left\{ |(Th, h)| : \text{for every unit vector } h \text{ in } H \right\} \]

The latter fact was discovered by C.A. Berger (not yet published).

Using function theoretic methods, B. Sz-Nagy and C. Foias have given a characterization of $C_\rho$ and shown the monotony of $C_\rho$ as a generalization of $C_1$ and $C_\alpha$. Hence we may naturally expect that the condition for $T \in C_\rho$ depends upon $\| T \|$ and $\| T \|_\alpha$ together. In this paper, as a continuation of calculations in the preceding paper [3], we give a simple sufficient condition for $T \in C_\rho$ related to both $\| T \|$ and $\| T \|_\alpha$ and its graphic expression.

1. The following theorems are known.

Theorem A ([5]). An operator $T$ in $H$ belongs to the class $C_\rho$ if and only if it satisfies the following conditions:

(i) $\| h \|^2 - 2 \left( 1 - \frac{1}{\rho} \right) \text{Re}(zTh, h) + \left( 1 - \frac{2}{\rho} \right) |zTh|^2 \geq 0$

for $h$ in $H$ and $|z| \geq 1$.

(ii) The spectrum of $T$ lies in the closed unit disk.

Theorem B ([5]). $C_\rho$ is non-decreasing with respect to the index $\rho$ in the sense that

$C_{\rho_1} \subseteq C_{\rho_2}$ if $0 \leq \rho_1 < \rho_2$.

Theorem C ([1]).
(i) \[
\begin{cases}
\text{If } \|T\| \leq \frac{\rho}{2-\rho} \text{ and } 0 \leq \rho \leq 1, \text{ then } T \in C_\rho. \\
\text{If } \|T\| \leq 1, \text{ then } T \in C_\rho \text{ for } \rho \geq 1.
\end{cases}
\]

(ii) \[
\begin{cases}
\text{If } T \in C_\rho \text{ for } 0 \leq \rho \leq 1, \text{ then } r(T) \leq \frac{\rho}{2-\rho}.
\end{cases}
\]

If \( T \in C_\rho \) for \( \rho \geq 1 \), then \( r(T) \leq 1 \).

where \( r(T) \) means the spectral radius of \( T \).

An operator \( T \) is called to be normaloid if \( \|T\| = \|T\|_N \) or equivalently the spectral radius is equal to \( \|T\|_N \).

**Theorem D** ([1][3]). If \( T \) is normaloid, \( T \in C_\rho \) if and only if
\[
\|T\| \leq \begin{cases}
\frac{\rho}{2-\rho} & \text{if } 0 \leq \rho \leq 1 \\
1 & \text{if } \rho \geq 1
\end{cases}
\]

Theorem D was proved by E. Durszt for normal operators and by C.A. Berger and J.G. Stampfli ([1]). The author has given a simplified proof of the same theorem in [3] independently.

2. For \( 0 \leq \rho \leq 2 \), the condition (I) is replaced by
\[
(2-\rho) \|z Th\|^p - 2(1-\rho) \text{Re}(z Th, h) - \rho \|h\|^p \leq 0 \quad \text{for } h \in H.
\]
That is,
\[
(2-\rho) \|z Th\|^p - 2(1-\rho) \text{Re}(z Th, h) - \rho \|h\|^p \leq (2-\rho) \|Th\|^p - 2(1-\rho) \|Th, h\| - \rho \|h\|^p \leq 0
\]
for every unit vector \( h \) in \( H \).

**Theorem 1.** (I) implies \( \|T\|_N \leq \begin{cases}
\frac{\rho}{2-\rho} & \text{if } 0 \leq \rho \leq 1 \\
1 & \text{if } 1 \leq \rho \leq 2.
\end{cases} \)

**Proof.** Let \( 0 \leq \rho \leq 1 \). By \( (I') \), \( (I) \) is equivalent to
\[
F_\rho(\rho, h) = (2-\rho) \|Th\|^p - 2(1-\rho) \|Th, h\| - \rho \|h\|^p \leq 0
\]
for every unit vector \( h \) in \( H \). That is
\( (I) \) is true if and only if \( \sup_{||h||=1} F_\rho(\rho, h) \leq 0 \).

The following inequality is clear
\[
(2-\rho) \|T\|^p + 2(1-\rho) \|T\|_N - \rho \leq \sup_{||h||=1} F_\rho(\rho, h) \leq (2-\rho) \|T\|^p + 2(1-\rho) \|T\|_N - \rho.
\]
Consequently \( (I) \) implies
\[
(2-\rho) \|T\|^p + 2(1-\rho) \|T\|_N - \rho \leq 0,
\]
\[
(||T||_N + 1) \cdot (2-\rho) \|T\|_N - \rho \leq 0.
\]
Hence
\[
\|T\|_N \leq \frac{\rho}{2-\rho}.
\]

Now let \( 1 \leq \rho \leq 2 \), then the condition \( (I'') \) is equivalent to
\[ F_z(\rho, h) = (2-\rho) \| Th \|^2 + 2(\rho - 1) \| (Th, h) \| - \rho \leq 0 \]

for every unit vector \( h \) in \( H \). That is

\[ (I_\rho) \text{ is true if and only if } \sup_{|h|=1} F_z(\rho, h) \leq 0. \]

The following inequality is also clear.

\[ (**) \quad (2-\rho) \| T \|_N^2 + 2(\rho - 1) \| T \|_N - \rho \leq \sup_{|h|=1} F_z(\rho, h) \leq (2-\rho) \| T \|_N - 2(\rho - 1) \| T \| - \rho. \]

Consequently (\( I_\rho \)) implies

\[ (2-\rho) \| T \|_N^2 + 2(\rho - 1) \| T \|_N - \rho \leq 0, \]

\[ (\| T \|_N - 1)(2-\rho) \| T \|_N + \rho \leq 0. \]

Hence

\[ \| T \|_N \leq 1 \]

q.e.d.

Theorem 1 gives a precise limitation of \( \| T \|_N \) for \( T \in C_\rho \). Since \( r(T) \leq \| T \|_N \) (4)) we get immediately.

Corollary 1 (\([5]\)). For \( \rho \leq 2 \), \( (I_\rho) \) implies (II).

C. A. Berger has characterized \( T \in C_\rho \) by \( \| T \|_N \leq 1 \). This fact and the monotony of \( C_\rho \) give the corollary 1. But in our method the estimation of \( \| T \|_N \) comes to give the proof without complicated calculations. Moreover by (\( * \)) and (\( ** \)) in the proof of Theorem 1 we can sharpen Theorem C and give a simple sufficient condition for \( T \in C_\rho \) as shown in the next section.

3. The following theorems are obvious by Theorem 1 and inequalities (\( * \)), (\( ** \)).

Theorem 2. (i) For \( 0 \leq \rho \leq 1 \). \( T \in C_\rho \) if and only if \( \sup_{|h|=1} F_z(\rho, h) \leq 0 \). (ii) For \( 1 \leq \rho \leq 2 \). \( T \in C_\rho \) if and only if \( \sup_{|h|=1} F_z(\rho, h) \leq 0 \).

Theorem 3. (i) For \( 0 \leq \rho \leq 1 \). If \( T \in C_\rho \), then \( \| T \|_N \leq \frac{\rho}{2-\rho} \).

(ii) For \( 1 \leq \rho \leq 2 \). If \( T \in C_\rho \), then \( \| T \|_N \leq 1 \).

Theorem 4. (i) For \( 0 \leq \rho \leq 1 \). If \( (2-\rho) \| T \|_N^2 + 2(1-\rho) \| T \|_N - \rho \leq 0 \), then \( T \in C_\rho \).

(ii) For \( 1 \leq \rho \leq 2 \). If \( (2-\rho) \| T \|_N^2 + 2(\rho - 1) \| T \|_N - \rho \leq 0 \), then \( T \in C_\rho \).

Corollary 2 ([1]). (i) For \( 0 \leq \rho \leq 1 \). If \( \| T \| \leq \frac{\rho}{2-\rho} \), then \( T \in C_\rho \).

(ii) For \( \rho \geq 1 \). If \( \| T \| \leq 1 \), then \( T \in C_\rho \).

Proof of Corollary 2. (ii) is clear and (i) is also derived from (i) of Theorem 4 replacing \( \| T \|_N \) by \( \| T \| \).

q.e.d.

Theorem 5. There exists \( k \) in \([1/2, 1]\) such that

(i) if \( T \in C_\rho \) for \( 0 \leq \rho \leq 1 \), then \( (2-\rho) \| T \|_N^2 + 2(1-\rho) \| T \|_N - \rho \leq 0 \).

(ii) if \( T \in C_\rho \) for \( 1 \leq \rho \leq 2 \), then \( (2-\rho) \| T \|_N^2 + 2(\rho - 1) \| T \|_N - \rho \leq 0 \).
Proof. Take sequences of unit vectors \( \{h_n\} \) in (\( *) \) and (\( ** \)) which \( |(Th_n, h_n)| \) converges to \( ||T||_\infty \), then \( ||T||_\infty \leq \sup ||Th_n|| \leq ||T|| \). By this inequality and \( 1/2||T|| \leq ||T||_\infty \leq ||T|| \), we get Theorem 5. q.e.d.

4. We consider an operator \( T \) which \( ||T|| \) and \( ||T||_\infty \) equal \( s \) and \( s' \) respectively. For example \( T_s = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \). We can show \( ||T_s|| = s, ||T_s||_\infty = s'/2 \) and \( r(T_s) = 0 \) by simple calculations. Then by Theorem 4 we know

\[
T \in \begin{cases} 
C_{\frac{s^2+s}{s^2+s+1}} & \text{if } 0 \leq s \leq 1 \\
C_{\frac{s^2-s}{s^2-s+1}} & \text{if } 1 \leq s \leq 2.
\end{cases}
\]

In [4] it is shown that \( T_s \in C_{s^2} \) if \( 0 \leq s \leq 1 \). But by our estimation we get more precisely

\[
T_s \in C_{\frac{s^2+s}{s^2+s+1}} \subset C_{\frac{2s}{s+1}}.
\]

However it is known by Durszt [2] that this operator belongs to more narrow class \( C_{*} \). On the other hand we get the following inequality by Theorem 3

\[
||T_s||_\infty = \frac{s}{s+1} \begin{cases} 
\frac{s}{s} & \text{if } 0 \leq s \leq 1 \\
\frac{s}{s} & \text{if } 1 \leq s \leq 2.
\end{cases}
\]

Thus we know Theorem 3 and 4 give sharpenings of Theorem C exactly.

5. Theorem 4 indicates a sufficient condition for \( T \in C_{\rho} (0 \leq \rho \leq 2) \) depending upon \( ||T|| \) and \( ||T||_\infty \) together. We can represent the relation among operator norm \( ||T|| \), numerical radius \( ||T||_\infty \) and this sufficient condition by a domain ODE or OAF in a triangle OAB in the figure below. The curves DE and AF are given by

\[
F_s(\rho) = (2-\rho) ||T||^2 + 2(1-\rho) ||T||_\infty - \rho = 0 \quad \text{for } 0 \leq \rho \leq 1
\]

\[
F_s(\rho) = (2-\rho) ||T||^2 + 2(\rho-1) ||T||_\infty - \rho = 0 \quad \text{for } 1 \leq \rho \leq 2
\]

respectively.

When \( \rho \to 1 \), \( F_s(\rho) \) and \( F_s(\rho) \) gradually close to \( ||T||^2 - 1 = 0 \) and the curves DE and AF close to the vertical line AC. Moreover \( F_s(\rho) \) passes \( A(1,1) \) for every \( \rho \) and when \( \rho \to 2 \), \( F_s(\rho) \) gradually close to \( ||T||_\infty - 1 = 0 \) and the curve AF closes to the horizontal line AB. The triangular domains OAC and OAB indicate the necessary and sufficient condition for \( T \) to belong to \( C_1 \) and \( C_2 \) respectively. The line OA indicates the degenerated domain which give the necessary and sufficient condition for a normaloid operator \( T \) to belong to \( C_{\rho} (0 \leq \rho \leq 1) \), where the coordinates of \( D \) are \( \left( \frac{\rho}{2-\rho}, \frac{\rho}{2-\rho} \right) \) by Theorem 4 and Theorem D.
References


