16. On Criterion for the Nuclearity of Space $S\{M_p\}$

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In his paper [2], T. Yamanaka introduced a new type of function spaces $S\{M_p\}$ which includes $K\{M_p\}$ as well as all S-type spaces. In this note, we shall consider a criterion for the nuclearity of the space $S\{M_p\}$. The fundamental idea of its proof is essentially due to [1], [3]. For nuclear spaces and its related notion, see [1].

Let $M_p(x, q)(p=1, 2, \cdots)$ be functions defined for all $x \in R_n$ (n-dimensional Euclidean space) and all systems of n non-negative integers $q=(q_1, q_2, \cdots, q_n)$ which satisfy the following three conditions.

(1) $0 \leq M_1(x, q) \leq M_2(x, q) \leq \cdots \leq M_p(x, q) \leq \cdots$

(2) For every p there exists a positive number N_p which may be infinite, such that $\lim_{p\to\infty} N_p = \infty$ and $\inf_x M_p(x, q) > 0$ for $|q| < N_p$ and $M_p(x, q) = 0$ for $|q| \geq N_p$.

(3) For any fixed pair (x, q) there are only two possible cases;

$M_p(x, q) = \infty$ for all p or $M_p(x, q) < \infty$ for all p.

Given such a system of functions $M_p(x, q)$, we denote by $S\{M_p\}$ the set of all infinitely differentiable functions $\varphi(x)$ for which the countable norms are finite, i.e.

$$\| \varphi \|_p = \sup_{x,q} M_p(x, q) | D^q \varphi(x) | < \infty.$$

Proposition 1. *The space $S\{M_p\}$ is complete.*

Proof of this proposition is found in ([2] or [3]).

We will say that a space $S\{M_p\}$ satisfies condition (N_1), if the following conditions hold.

(1) For any p there is $p' \geqq p$ such that the ratio

$$m_{pp'}(x) = \sup_q \frac{M_p(x, q)}{M_{p'}(x, q)} \qquad \left(\frac{0}{0} = \frac{\infty}{\infty} = 0\right).$$

goes to zero as $|x| \to \infty$ and $m_{pp'}(x)$ is a summable function of x.

(2) If there exists q such that $M_p(x,q) \neq 0, \neq \infty$ for every $x \in R_n$, then we can obtain the following inequality:

$$M_p(x, q) \leq K_{pp'} M_{p'}(y, q+\alpha) \quad \text{for} \quad |y-x| \leq 1 \text{ and } |\alpha| \leq n,$$

where $K_{pp'}$ is a suitable constant number and n is a arbitrary positive integer. The following Lemma is due to [3]:

Lemma. *Let $\varphi(x)$ be a n-ordered continuous differentiable function on $B(x; r)$,*[1] *then we can obtain the following inequality* $|\varphi(x)| \leq A_r \sum_{|\beta| \leq n} \int_{|y-x| \leq r} |D^\beta \varphi(y)| dy$, *where A_r is a suitable constant*

1) $B(x; r)$ denotes the closed ball with center x and radius r.

number which does not depend on $\varphi(x)$.

Theorem 1. *If the space* $S\{M_p\}$ *satisfies condition* (N_1), *then it is a countable Hilbert space.*

Proof. We introduce in $S\{M_p\}$ a countable collection of scalar products setting

$$(\varphi, \psi)_p = \sup_q \int [M_p(x, q)]^2 \sum_{0 \le r \le q} D^r\varphi(x) D^r\psi(x) dx.$$

We shall show that the topology in $S\{M_p\}$, defined by the norms $\|\varphi\|_p$, concides with the topology defined by the norms $\|\varphi\|'_p = \sqrt{(\varphi, \varphi)_p}$. In fact

$$\begin{aligned}&[M_p(x, q)]^2 \,|\, D^q\varphi(x) \,|^2 \\ &= \left(\frac{M_p(x, q)}{M_{p'}(x, q)}\right)^2 [M_{p'}(x, q)]^2 \,|\, D^q\varphi(x)\,|^2 \\ &\le \left(\sup_q \frac{M_p(x, q)}{M_{p'}(x, q)}\right)^2 \sup_{x,q} \{[M_p(x, q)]^2 \,|\, D^q\varphi(x)\,|^2\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup_q \int [M_p(x, q)]^2 \,|\, D^q\varphi(x)\,|^2 dx \\ &\qquad \le \sup_{q,x} \{M_p(x, q)^2 \,|\, D^q\varphi(x)\,|^2\} \int m^2_{pp'}(x) dx.\end{aligned}$$

But by condition (N_1), p' can be chosen such that the integral $\int \sup_q \left(\frac{M_p(x, q)}{M_{p'}(x, q)}\right) dx = \int m_{pp'}(x) dx$ converges. Since $\lim_{|x|\to\infty} m_{pp'}(x) = 0$, the integral $\int m^2_{pp'}(x) dx$ converges. We denote its value by $B^2_{pp'}$. Then we obtain

$$\sup_q \int [M_p(x, q)]^2 \,|\, D^q\varphi(x)\,|^2 dx \leqq B^2_{pp'} \,\|\varphi\|^2_{p'},$$

and

$$\|\varphi\|'^2_2 = \sup_q \int [M_p(x, q)]^2 \,|\, D^q\varphi(x)\,|^2 dx \le C^2 \,\|\varphi\|^2_{p'},$$

i.e., $\|\varphi\|'^2_p < C^2 \|\varphi\|^2_{p'}$, where C is a constant not depending upon $\varphi(x)$.

Conversely, for x satisfying $M_p(x, q) \ne 0, \ne \infty$, by Lemma and condition (N_1) we have the following inequality:

$$\begin{aligned}&[M_p(x, q)]^2 \,|\, D^q\varphi(x)\,|^2 \leqq A^2_p [M_p(x, q)]^2 \sum_{|\beta| \le n} \int_{|y-x| \le 1} [D^{q+\beta}\varphi(y)]^2 dy \\ &\leqq A^2_p K^2_{pp'} \sum_{|\beta| \le n} \int_{|y-x \;\le 1} [M_{p'}(y, q+\beta)]^2 |\, D^{q+\beta}\varphi(y)\,|^2 dy \quad \text{for all } \varphi(x) \in S\{M_p\},\end{aligned}$$

where A_p is a suitable constant number which is not depending upon $\varphi(x)$. From this inequality we have

$$\begin{aligned}&\sup_{x,q} [M_p(x, q)]^2 \,|\, D^q\varphi(x)\,|^2 \\ &\qquad \le C \sup_q \sum_{|\beta| \le n} \int_{|y-x| \le 1} [M_{p'}(y, q+\beta)]^2 \,|\, D^{q+\beta}\varphi(y)\,|^2 dy,\end{aligned}$$

i.e., $\|\varphi\|_p^2 \leqq C \|\varphi\|_{p'}^{\prime 2}$, where C is a constant not depending upon $\varphi(x)$. Moreover, for x such that $M_p(x,q)=0, =\infty$, the same inequality is true. Therefore, for every $x \in R_n$, we obtain the inequality $\|\varphi\|_p^2 \leqq C \|\varphi\|_{p'}^{\prime 2}$.

It follows that the system of norms $\|\varphi\|_p'$ defines the same topology in $S\{M_p\}$ as does the system of norms $\|\varphi\|_p$. Thus if condition (N_1) is fulfilled, then $S\{M_p\}$ is a countably Hilbert space.

In conclusion, we consider some conditions under which a countable Hilbert space $S\{M_p\}$ is a nuclear space. We will say that a space $S\{M_p\}$ satisfies condition (N_2),

(1) The functions $M_p(x,q)$ are monotonically increasing for $|x|$.

(2) For any p there are p' and a positive constant number C such that $M_p(x\pm 1, q) \leq CM_{p'}(x, q)$.

The following proposition is due to [1]:

Proposition 2. *The space $\widetilde{K(a)}$ of all infinitely differentiable functions on the interval $[-a, a]$ is nuclear.*

Now, the space H is a nuclear space and the matrix $M=\|m_{n,p}\|$ has the following properties,

(1) $0 < m_{n,p} \leq m_{n+1,p}$ and $m_{n,p} \leq m_{n,p+1}$.

(2) For any k there exists p such that the series $\sum_{n=1}^{\infty} m_{n,k}/m_{n,p}$ converges, and its terms are monotonically decreasing.

We denote by $H(M)$ the set of all $\hat{\varphi}=(\varphi_1, \varphi_2, \cdots) \in H \times H \times \cdots$ such that $\|\hat{\varphi}\|_p^2 = \sum_{n=1}^{\infty} m_{n,p} \|\varphi_n\|_p^2$ converges.

Then we obtain following:

Proposition 3. *The space $H(M)$ is a nuclear space.*

Proof. First we prove that the space $H(M)$ is a countable Hilbert space. We introduce in $H(M)$ a countable collection of scalar products $(\hat{\varphi}, \hat{\psi})_p = \sum_{n=1}^{\infty} M_{n,p}(\varphi_n, \psi_n)_p$, then $H(M)$ is a countably Hilbert space. Let us now prove that $H(M)$ is a nuclear space. It is sufficient that for any k there exists p such that an imbedding operator T_k^p; $H(M_p)$[2]$\to H(M)_k$ is Hilbert-Schmidt type. Since H is a nuclear space, for any k there exists n such that an imbedding operator t_k^n; $H_n \to H_k$ is Hilbert-Schmidt type. Moreover, $\sum_{l=1}^{\infty} m_{l,k}/m_{l,n} < \infty$. Therefore, let $\{h_i\}$ be an orthonormal basis then $\sum_{i=1}^{\infty} \|h_i\|_k^2 < \infty$. Now, let $\hat{h}_{l,i} = (0, 0, \cdots, 0, \underset{(l\text{-th})}{h_i/\sqrt{m_{l,n}}}, 0, \cdots, 0)$. Since $\|h_i\|=1$, we obtain

$$\|\hat{h}_{l,i}\|^2 = m_{l,n} \|h_i/\sqrt{m_{l,n}}\|^2 = 1 \text{ and } (\hat{h}_{l,i}, \hat{h}_{t,s})$$

$$= \begin{cases} m_{l,n}(h_i, h_s)_n & \text{for the case } \begin{pmatrix} t=l \\ i=s \end{pmatrix}, \\ 0 & \text{for the other case.} \end{cases}$$

2) The Hilbert space $H(M)_p$ is the completion of the space $H(M)$ by the norms $\|\hat{\varphi}\|_p = \sqrt{(\hat{\varphi}, \hat{\varphi})_p}$.

Therefore, $\{\hat{h}_{l,i};\, l,\, i\}$ is an orthonormal basis in $H_n(M)$. Since

$$\| \hat{h}_{l,i} \|_k = m_{l,k} \| h_i/\sqrt{m_{l,n}} \|_k^2 = m_{l,k}/m_{l,i} \| h_i \|_k^2,$$

we have the following equality:

$$\sum_{l,i} \| \hat{h}_{l,i} \|_k^2 = \sum_{l,i} m_{l,k}/m_{l,n} \| h_i \|_k^2 = \sum_l m_{l,k}/m_{l,n} \sum_i \| h_i \|_k^2.$$

Hence, an imbedding operator is Hilbert-Schmidt type. The proof is complete.

Combining Proposition 3 with Proposition 2, we have the following.

Proposition 4. *If the matrix* $M = \| m_{n,p} \|$ *has the properties indicated above, then the space* $\widetilde{K(M)}$ *consisting of functions* $\varphi(x)(-\infty < x < \infty)$ *infinitely differentiable on every interval* $[n, n+1]$, *having one sided derivatives of all orders at every integer point, and such that for any* p *the series* $\sum_{n=-\infty}^{\infty} m_{n,p} \int_n^{n+1} | D^q \varphi(x) |^2 dx (0 \leq q \leq p)$ *converge, is nuclear.*

Now, we define the system of norms

$$||| \varphi |||_p^2 = \sum_{0 \leq r \leq q} \sum_{n=-\infty}^{\infty} m_{n,p} \int_n^{n+1} | D^r \varphi(x) |^2 dx \quad \text{for } \varphi(x) \in \widetilde{K(M)},$$

where we have put $m_{n,p} = \sup_{n \leq x \leq n+1} M_p^2(x, q)$.

By the condition (N_1), (N_2) on the $M_p(x, q)$, the system of numbers $m_{n,p}$ has the properties formulated earlier. Therefore the space $\widetilde{K(M)}$ corresponding to the system $M = \| m_{n,p} \|$ is nuclear by Proposition 4. On the other hand, if the space $S\{M_p\}$ satisfies condition (N_2), then the system of norms $\| \varphi \|_p'^2 = \sup_q \int [M_p(x, q)]^2 | D^q \varphi(x) |^2 dx$ is equivalent to the system of norms $||| \varphi |||_p^2$ by definition of $m_{n,p}$. But the space $S\{M_p\}$ is a closed subspace of $\widetilde{K(M)}$ by Proposition 1. Therefore, we obtain from the result just proven the following

Theorem 2. *If the space* $S\{M_p\}$ *satisfies condition* (N_1) *and* (N_2), *then it is a nuclear space.*

References

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[3] ——: Topological Linear Space and Generalized Functions (in Japanese) Kyoritsu, Tokyo (1966).