12. *Note on the Nuclearity of Some Function Spaces. I*

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The definition of nuclearity in a general locally convex space was first given by A. Grothendieck [4]. The definition of nuclearity given by M. Gelfand and N. Ya. Vilenkin [3] concides with that of [4] in the case of countably normed spaces.

In this note, we consider the condition for nuclearity in A. Pietsch [6], which is mainly derived from A. Grothendieck. By using its condition, we shall show that $K_\Omega\{M_A\}$ space indroduced first by I. M. Gelfand and G. E. Shilov [2] and extended by T. Yamanaka [7] is nuclear.

1. Let E be a locally convex Hausdroff space over real or complex fields and U is any absorbent and absolutely convex neighborhood of the origin in E. Let

$$p_U(x)=\inf\{\rho>0;\, x\in\rho U\} \text{ for } x\in E$$

and
$$E_U=E/\{x\in E;\, p_U(x)=0\},$$

then topology of E_U is introduced by the norm

$$\|x_U\|=p_U(x) \quad \text{for } x_U\in E_U$$

where x_U coresponds to $x\in E$ in a natural way.

Let $C(M)$ be the sets of all continuous real or complex valued functions defined on M which is a compact Hausdroff space. Each continuous linear from μ on $C(M)$ is called *a Radon measure* on M and we frequently writes

$$\mu(f)=\int_M fd\mu.$$

A "*positive*" Radon measure is a $\mu\in C(M)'$ such that $\mu(f)\geqq 0$ whenever $f(x)\geqq 0$ for all $x\in M$.

Let E and F be normed spaces and their closed unit balls be U and V respectively. A continuous linear mapping T of E in F is called *nuclear mapping* if there exists continuous linear form $a_n\in E'$ and $y_n\in F$ such that the following holds:

$$Tx=\sum_N\langle x, a_n\rangle y_n \quad \text{for } x\in E$$

and
$$\sum_N P_{U^0}(a_n)P_V(y_n)<+\infty.$$

Definition. A locally convex Hausdroff space E be called *nuclear space* when there eixsts a base $\mathcal{U}(E)$ of absolutely convex, absorbent 0-neighborhood such that the following equivalent conditions holds:

i) for any $U\in\mathcal{U}(E)$ there exists a $V\in\mathcal{U}(E)$ being absorbed by U such that the canonical mapping from E_V on E_U is nuclear.

ii) for any $U \in \mathcal{U}(E)$, there exists a $V \in \mathcal{U}(E)$ being absorbed by U such that the canonical mapping from E'_{U^0} in E'_{V^0} is nuclear. We need the following theorem due to A. Pietsch.

Theorem 1. *A locally convex Hausdroff space E is nuclear if and only if there exists a base $\mathcal{U}(E)$ of 0-neighborhood in E such that the following holds:*

(N) *for any $U \in \mathcal{U}(E)$ there exists a $V \in \mathcal{U}(E)$ and a positive Radon measure μ defined on the weakly compact polar V^0 such that*

$$p_U(x) \leqq \int_{V^0} |\langle x, a\rangle| \, d\mu \quad \text{for } x \in E.$$

The proof is given in [6].

2. $K_\Omega\{M_A\}$ space and it's nuclearity.

Let Ω be a open set in R^n, $x=(x_1, x_2, \cdots, x_n)$, $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ be variable points in Ω and $|x|=\left(\sum_{j=1}^n x_j^2\right)^{1/2}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ where $D_j = \partial/\partial x_j$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. $\alpha \geqq \beta$ means $\alpha_j \geqq \beta_j$ for $j=1, 2, \cdots, n$ and $\frac{a}{b} = \left(\frac{a_1}{b_1}, \cdots, \frac{a_n}{b_n}\right)$, $\frac{ak}{b} = \left(\frac{a_1k_1}{b_1}, \cdots, \frac{a_nk_n}{b_n}\right)$ where $a=(a_1, \cdots, a_n)$, $b=(b_1, \cdots, b_n)$, $k=(k_1, \cdots, k_n)$ and we obey the rule $0 \cdot \infty = \infty \cdot 0 = 0$, $\frac{\infty}{\infty} = \frac{0}{0} = 0$.

Definition. Let A be any directed index set. We assume that $M_p(x, q) (p \in A)$ is measurable on Ω with respect to x for each multi-index q and satisfies the following two conditions:

(i) $M_p(x, q) \geqq 0$ for any p in A, and if $p \leqq p'$, then $M_p(x, q) \leqq M_{p'}(x, q)$

(ii) for each $p \in A$ and multi-index q', there exists a constant C depending on p, q', and p' such that the inequality

$$M_p(x, q) \leqq C M_{p'}(x, q+q') \tag{1}$$

holds for all multi-index q.

Next, we put

$$\|\varphi\|_p = \sup\{M_p(x, q) \,|\, D^q \varphi(x)| \,|\, x \in \Omega, q; \text{multi-index}\}, \tag{2}$$

where φ is any infinitely differentiable function. Then denote by $K_\Omega\{M_A\}$ sets of all infinitely differentiable functions φ which satisfies $\|\varphi\|_p < +\infty$ for all $p \in A$, and topology of $K_\Omega\{M_A\}$ be defined by the sequence of semi-norm $\|\varphi\|_p (p \in A)$.

Here, we make the following three assumptions on the $K_\Omega\{M_A\}$:

(P) for any p in A there exists $p' > p$ such that to any $\varepsilon > 0$ there corresponds some $N_0 > 0$ such that if $|q| > N_0$ then

$$M_p(x, q) \leqq \varepsilon M_{p'}(x, q) \tag{3}$$

(N_1) for any p in A there exists $p' \geqq p$ such that

$$m_{pp'}(x)=\sup_q \frac{M_p(x,q)}{M_{p'}(x,q)} \tag{4}$$

is integrable on Ω.

(N_2) let us denote by Ω_{M_p} the sets of points (in Ω) where the $M_p(x, q)$ is not equal to zero and ∞ for some q and assume that for each $p \in A$ there exists $\gamma_p > 0$ such that $\{\xi \mid |\xi - x| \leqq \gamma_P\} \subset \Omega$ for all $x \in \Omega_{M_p}$, then

(1) for any $p \in A$ there exists $p' \geqq p$ and $K_{pp'} > 0$ such that for each $x \in \Omega_M$ if $|y-x| \leqq \gamma_P$ and $|q'| \leqq n$ then

$$M_p(x, q) \leqq K_{pp'} M_{p'}(y, q+q') \tag{5}$$

or

(2) $M_p(x, q)(p \in A)$ are monotone increasing in Ω with respect to $x \geqq 0$ and monotone decreasing in Ω with respect to $x < 0$.

Lemma 1. *If for any p in A, there exist a non-negative integer n_0, $p' \geqq p$ and constant $C = C_{pp'}$ such that the following inequality holds:*

$$\|\varphi\|_p \leqq C \sum_{0 \leqq |q| \leqq n_0} \int_\Omega M_{p'}(x, q) \, |D^q \varphi(x)| \, dx < +\infty (\varphi \in K_\Omega\{M_A\}) \tag{6}$$

then $K_\Omega\{M_A\}$ is a nuclear space.

Proof. Since the continuous linear forms δ_ξ^q defined by

$$\langle \varphi, \delta_\xi^q \rangle = M_{p'}(\xi, q) D^q \varphi(\xi) \quad \text{for } \xi \in \Omega,\ 0 \leqq |q| \leqq n_0 \tag{7}$$

be contained in the polar of the 0-neighborhood

$$V = \{\varphi \mid \varphi \in K_\Omega\{M_A\},\ \|\varphi\|_{p'} \leqq 1\}, \tag{8}$$

we can define a positive Radon measure μ on V^0 by the following equality:

$$\int_{V^0} \Phi(a) d\mu = C \sum_{0 \leqq |q| \leqq n_0} \int_\Omega \Phi(\delta_\xi^q) d\xi \quad \text{for } \Phi \in C(V^0) \tag{9}$$

therefore

$$\|\varphi\|_p \leqq \int_{V^0} |\langle \varphi, a \rangle| \, d\mu \quad \text{for all } \varphi \in K_\Omega\{M_A\}. \tag{10}$$

Hence, by Theorem 1, $K_\Omega\{M_A\}$ is a nuclear space.

Lemma 2. *For sufficiently small positive number ε and γ the following inequality holds:*

$$|\varphi(x)| \leqq A_r \sum_{|q| \leqq n} \int_{|\xi - x| \leqq \gamma} |D^q \varphi(\xi)| \, d\xi \tag{11}$$

or

$$|\varphi(x)| \leqq B_r \sum_{|q| \leqq n} \int_x^{x+\varepsilon} |D^q \varphi(\xi)| \, d\xi \tag{12}$$

and

$$|\varphi(x)| \leqq B'_r \sum_{|q| \leqq n} \int_{x-\varepsilon}^{x} |D^q \varphi(\xi)| \, d\xi \tag{12'}$$

where $\varphi \in C^\infty(\Omega)$, A_r, B_r, and B'_r are independent of φ.

Proof. Let $r(t)$ (t real) be a continuous differentiable function which equal 1 at $t=0$ and 0 for $|t| \geqq \varepsilon_1$, where ε_1 is a fixed positive number. Since

$$
\begin{aligned}
-\varphi(x) &= \gamma(\varepsilon_1)\varphi(x_1+\varepsilon_1, x_2, \cdots, x_n) - \gamma(0)\varphi(x) \\
&= \int_{x_1}^{x_1+\varepsilon_1} \frac{\partial}{\partial \xi_1} [\gamma(\xi_1 - x_1)\varphi(\xi_1, x_2, \cdots, x_n)] d\xi_1 \\
&= \int_{x_1}^{x_1+\varepsilon_1} \left(\frac{\partial \gamma}{\partial \xi_1}\varphi + \gamma \frac{\partial \varphi}{\partial \xi_1} \right) d\xi_1
\end{aligned}
$$

therefore we have

$$
|\varphi(x)| \leqq B\int_{x_1}^{x_1+\varepsilon_1} |\varphi(\xi_1, x_2, \cdots, x_n)| \, d\xi_1 + C\int_{x_1}^{x_1+\varepsilon_1} \left| \frac{\partial}{\partial \xi_1}\varphi(\xi_1, x_2, \cdots, x_n) \right| d\xi_1.
$$

With ε_1, x_1, $\gamma(x_1)$ replaced by $\varepsilon_2 > 0$, x_2, $\gamma(x_2)$, applying the same argument to

$$
\varphi(\xi_1, x_2, \cdots, x_n), \frac{\partial \varphi(\xi_1, x_2, \cdots, x_n)}{\partial \xi_1}
$$

and proceeding in this way step by step, we arrive at (12) and similarly (12)′, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$, then (11) provided we take $\varepsilon_0\sqrt{n} < \gamma$, where $\varepsilon_0 = max(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$.

Theorem 2. *If the space $K_\Omega\{M_A\}$ satisfies conditions (P), (N_1), and (N_2), then it is a nuclear space.*

Proof. For any $p \in A$ there exists $p' \geqq p$ such that $m_{pp'}(x)$ is integrable on Ω. (by (N_1)). Hence if $\varphi \in K_\Omega\{M_A\}$ then

$$
\begin{aligned}
M_p(x, q) \, |D^q\varphi(x)| &\leqq m_{pp'}(x) M_{p'}(x, q) \, |D^q\varphi(x)| \\
&\leqq m_{pp'}(x) \sup_{x \in \Omega} M_{p'}(x, q) \, |D^q\varphi(x)| \quad \text{for all } x \in \Omega.
\end{aligned}
$$

By integration

$$
\sup_q \int_\Omega M_p(x, q) \, |D^q\varphi(x)| \, dx \leqq \|\varphi\|_{p'} \left(\int_\Omega m_{pp'}(x) dx \right) < +\infty. \tag{13}
$$

Next, noting that if (P) holds then for all

$$
\lim_{|q| \to +\infty} \sup_x M_p(x, q) \, |D^q\varphi(x)| = 0 \tag{14}
$$

we have the equality (for some positive integer n_0)

$$
\|\varphi\|_p = \sup_{x \; q} \{ M_p(x, q) \, |D^q\varphi(x)| \mid x \in \Omega, 0 \leqq |q| \leqq n_0 \} (\varphi \in K_\Omega\{M_A\}. \tag{15}
$$

In the first place if we assume (N_2) (1), by (11) and (15), we have, for $\varphi \in K_\Omega\{M_A\}$ and $x \in \Omega_{M_p}$,

$$
\begin{aligned}
M_p(x, q) \, |D^q\varphi(x)| &\leqq A_{r_p} M_p(x, q) \sum_{|q'| \leqq n} \int_{|\xi - x| \leqq r_p} |D^{q+q'}\varphi(\xi)| \, d\xi \\
&\leqq A_{r_p} \cdot K_{pp'} \sum_{|q'| \leqq n} \int_{|\xi - x| \leqq r_p} M_{p'}(\xi, q+q') \, |D^{q+q'}\varphi(\xi)| \, d\xi,
\end{aligned}
$$

hence
$$
\|\varphi\|_p \leqq D_{pp'} \sup_{|q+q'| \leqq n_0} \int_\Omega M_{p'}(\xi, q+q') \, |D^{q+q'}\varphi(\xi)| \, d\xi
$$

i.e.
$$
\|\varphi\|_p \leqq D_{pp'} \sum_{0 \leqq |q''| \leqq n_0} \int_\Omega M_{p'}(x, q'') \, |D^{q''}\varphi(x)| \, dx < +\infty \tag{16}
$$

Next, if we assume (N_2) (2) and $x \geqq 0$, then, by (1), (12), and (15), we have for $\varphi \in K_\Omega\{M_A\}$, $x \in \Omega_{M_p}$,

$$M_p(x, q)\,|\,D^q\varphi(x)\,|\leqq B_r \sum_{|q'|\leqq n}\int_x^{x+\varepsilon} M_p(\xi, q)\,|\,D^{q+q'}\varphi(\xi)\,|\,d\xi$$
$$\leqq CB_{r_p}\sum_{|q'|\leqq n}\int_x^{x+\varepsilon} M_{p'}(\xi, q+q')\,|\,D^{q+q'}\varphi(\xi)\,|\,d\xi,$$

hence $$\|\varphi\|_p\leqq C_{pp'}\sum_{0\leqq|q''|\leqq n_0}\int_\Omega M_{p'}(x, q'')\,|\,D^{q''}\varphi(x)\,|\,dx<+\infty. \qquad (17)$$

In the case of $x<0$, it is quite similar by using (12)'. Therefore, by Lemma 1, $K_\Omega\{M_A\}$ is nuclear.

Remark. It will be found with its proof in [1] or [2] what we stated without proof in 2.

References

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