10. *Simple Type Theory of Gentzen Style with the Inference of Extensionality*

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The definitions of types and expressions are the same as in [2]. We also refer to expressions of type 1 as formulas. As in *LK* or *GLC*, the form

$$A_1, \cdots, A_m \rightarrow B_1, \cdots, B_n,$$

where $A_1, \cdots, A_m, B_1, \cdots, B_n (m, n \geqq 0)$ are formulas, is called a sequent.

The inference rules of our system are as follows:

(I) Structual inference rules

$$\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}$$

$$\frac{A, A, \Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} \qquad \frac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A}$$

$$\frac{\Gamma, A, B, \Pi \rightarrow \Delta}{\Gamma, B, A, \Pi \rightarrow \Delta} \qquad \frac{\Gamma \rightarrow \Delta, A, B, \Lambda}{\Gamma \rightarrow \Delta, B, A, \Lambda}$$

(Cut) $$\frac{\Gamma \rightarrow \Delta, A \quad A, \Pi \rightarrow \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda}$$

(II) Inference rules on logical symbols

$$\frac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta} \qquad \frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}$$

$$\frac{A, \Gamma \rightarrow \Delta \quad B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta}$$

$$\frac{\Gamma \rightarrow \Delta, A}{\Gamma \rightarrow \Delta, A \vee B} \qquad \frac{\Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \vee B}$$

$$\frac{A(a^\tau), \Gamma \rightarrow \Delta}{\exists x^\tau A(x^\tau), \Gamma \rightarrow \Delta}, \qquad \frac{\Gamma \rightarrow \Delta, A(e^\tau)}{\Gamma \rightarrow \Delta, \exists x^\tau A(x^\tau)},$$

where a^τ does not occur in the lower sequent. where e^τ is an arbitrary expression of type τ.

(III) Inference of comprehension

$$\frac{A(e_1^{\tau_1}, \cdots, e_n^{\tau_n}), \Gamma \rightarrow \Delta}{(e_1^{\tau_1}, \cdots, e_n^{\tau_n} \in \lambda x_1^{\tau_1} \cdots x_n^{\tau_n} A(x_1^{\tau_1}, \cdots, x_n^{\tau_n})), \Gamma \rightarrow \Delta}$$

$$\frac{\Gamma \rightarrow \Delta, A(e_1^{\tau_1}, \cdots, e_n^{\tau_n})}{\Gamma \rightarrow \Delta, (e_1^{\tau_1}, \cdots, e_n^{\tau_n} \in \lambda x_1^{\tau_1} \cdots x_n^{\tau_n} A(x_1^{\tau_1}, \cdots, x_n^{\tau_n}))}$$

(IV) Inference of extensionality

$$\frac{S_{1i_1} \cdots S_{1i_m} \quad S_{2i_1} \cdots S_{2i_m}}{(d_1^{\tau_1}, \cdots, d_n^{\tau_n} \in e(^{(\tau_1, \cdots, \tau_n)}), \Gamma \longrightarrow \varDelta, (e_1^{\tau_1}, \cdots, e_n^{\tau_n} \in e^{(\tau_1, \cdots, \tau_n)})},$$

where

1) $e^{(\tau_1, \cdots, \tau_n)}$ is a free variable or a constant;
2) at least one of $\tau_1, \cdots, \tau_n$ is $\neq 0$;
3) $\tau_i = 0$ implies $d_i^{\tau_i} = e_i^{\tau_i}$;
4) $i_1, \cdots, i_m$ are all the indices i with $\tau_i \neq 0$;
5) if $\tau_i = 1$, the S_{1i} and S_{2i} denote the sequents

$$d_i^{\tau_i}, \Gamma \longrightarrow \varDelta, e_i^{\tau_i}$$
$$e_i^{\tau_i}, \Gamma \longrightarrow \varDelta, d_i^{\tau_i}$$

respectively;

6) if $\tau_i = (\sigma_{i1}, \cdots, \sigma_{ir})$, then S_{1i} and S_{2i} denote the sequents

$$(a_{i1}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}} \in d_i^{\tau_i}), \Gamma \longrightarrow \varDelta, (a_{iq}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}} \in e_i^{\tau_i}),$$
$$(a_{i1}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}} \in e_i^{\tau_i}), \Gamma \longrightarrow \varDelta, (a_{i1}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}} \in d_i^{\tau_i})$$

respectively ($a_{i1}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}}$ should not occur in the lower sequent of this inference).

Definition of proof-figure

(1) $A \longrightarrow A$ is a proof-figure with the end-sequent $A \longrightarrow A$.

(2) If

$$\frac{S_1 \cdots S_k}{S}$$

is an instance of one of the inference rules stated above and $P_1, \cdots, P_k$ are proof-figures with the end-sequents $S_1, \cdots, S_k$ respectively, then

$$\frac{P_1 \cdots P_k}{S}$$

is a proof-figure with the end-sequent S.

A sequent S is said to be provable if there exists a proof-figure with the end-sequent S.

Similarly we can define the notions of "proof-figure without cut" and "provable without cut".

It can be verified that our system is essentially equivalent to the usual system of simple type theory with the axiom of extensionality but without axiom of choice.

Now the cut-elimination theorem holds in our system, viz.

Theorem. *Every provable sequent is provable without cut.*

The proof is based on the method used in [3]. First of all, we modify Schütte's notion of "semi-valuation" to adapt it to our system.

The differences are the following two points.

1) If $\exists x^\tau A(x^\tau)$ is t in a semi-valuation (in our sence) then there exists a free variable a^τ such that $A(a^\tau)$ is t in the valuation.

2) Suppose that

(1) $(d_1^{\tau_1}, \cdots, d_n^{\tau_n} \in e^{(\tau_1, \cdots, \tau_n)})$ is t,

(2) $(e_1^{\tau_1}, \cdots, e_n^{\tau_n} \in e^{(\tau_1,\cdots,\tau_n)})$ is f,

(3) $e^{(\tau_1,\cdots,\tau_n)}$ is a free variable or a constant and

(4) $d_i^{\tau_i} = e_i^{\tau_i}$, whenever $\tau_i = 0$.

Then a semi-valuation in our sence requires that there exists an $i (1 \leqq i \leqq n)$ such that either

(i) $\tau_i = 1$ and one of the formulas d_i^1, e_i^1 is t and the other is f, or

(ii) $\tau_i = (\sigma_{i1}, \cdots, \sigma_{ir})$ and, for some free variables $a_{i1}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}}$, one of the formulas $(a_{i1}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}} \in d_i^{\tau_i})$, $(a_{i1}^{\sigma_{i1}}, \cdots, a_{ir}^{\sigma_{ir}} \in e_i^{\tau_i})$ is t and the other is f. Then, similarly to [2] we can prove

(I) *If a sequent $A_1, \cdots, A_m \rightarrow B_1, \cdots, B_n$ is not provable without cut, then there exists a semi-valuation in which A_i is t for all $i = 1, \cdots, m$ and B_j is f for all $j = 1, \cdots, n$.*

Next, we have

(II) *For any given semi-valuation V, we can construct a general model $\mathfrak{M}_V$ and an assignment φ_V such that, whenever a formula A is t (or f) in V, A holds (or does not hold) in $\mathfrak{M}_V$ by the assignment φ_V, respectively.*

The construction proceeds by the induction on types.

On the other hand, we can also prove

(III) *Let $\mathfrak{M}$ be a general model and φ be an assignment. If a sequent $A_1, \cdots, A_m \rightarrow B_1, \cdots, B_n$ is provable, then some A_i does not hold in $\mathfrak{M}$ by φ or else some B_j holds in $\mathfrak{M}$ by φ.*

The cut-elimination theorem immediately follows from (I), (II), and (III).

References

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[3] M. Takahashi: A proof of cut-elimination theorem in simple type theory. J. Math. Soc. Japan, **19**, 399–410 (1967).