36. *On an Analytic Index-formula for Elliptic Operators*

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§ 1. **Preliminaries.** In his work [1], [3], M. F. Atiyah indicated an analytic formula for the index of elliptic differential operators on compact manifolds. The aim of this note is to describe this formula more explicitly.

Assume that both X and Y are differentiable vector bundles with fibre C^l over a compact oriented Riemannian manifold M without boundary and that they are provided with hermitian metric in each fibre. Let P be an elliptic differential operator of order m from $\mathcal{E}(X)$ to $\mathcal{E}(Y)$, where $\mathcal{E}(X)$ is the space of C^∞ sections of X provided with the usual topology. We denote by $L^2(X)$ the space of L^2 sections of X. Then, considered as a densely defined linear operator from $L^2(X)$ to $L^2(Y)$, P is closable. We denote its minimal closed extension by the same symbol P. Since P is a densely defined closed operator, there is its adjoint P^* which is a densely defined closed operator from $L^2(Y)$ to $L^2(X)$. It is well known that P has a finite index $\mathrm{Ind}\,(P)$.

§ 2. **Results.** Our first result is the following:

Theorem 1. *Let λ be a positive number. Then we have the formula*

(1) $$\mathrm{Ind}\,(P)=\lim_{\lambda\to\infty}\lambda[\mathrm{Trace}\,(\lambda+(P^*P)^k)^{-1}-\mathrm{Trace}\,(\lambda+(PP^*)^{k-1}]$$

where k is an arbitrary integer which is larger than $\frac{n}{2m}$.

Proof. The following proof is a variant of the discussion used in M. F. Atiyah and R. Bott [3].

Let $\Lambda=\{0, \lambda_1, \lambda_2, \cdots\}$ be the set of eigen values of PP^* or P^*P with $0<\lambda_1<\lambda_2<\cdots$. Let $\Gamma_j(X)$ and $\Gamma_j(Y)$ be, respectively, the eigen-spaces of P^*P and PP^* corresponding to λ_j. It is well known that $\Gamma_j(X), \Gamma_j(Y)$ are of finite dimension. Let P_j denote the restriction of P to $\Gamma_j(X)$. Then we have the following complexes:

$$0\longrightarrow\Gamma_j(X)\xrightarrow{P_j}\Gamma_j(Y)\longrightarrow 0,\qquad j=0,1,2,3,\cdots.$$

Obviously,

$$\mathrm{Ind}\,(P)=\dim\Gamma_0(X)-\dim\Gamma_0(Y),$$
$$0=\dim\ker P_j-\dim\mathrm{coker}\,P_j,$$

because $P^*P\,|_{\Gamma_j(X)}=\lambda_j$, $PP^*\,|_{\Gamma_j(Y)}=\lambda_j$. Hence

$$\mathrm{Ind}\,(P)=\sum_j \frac{\lambda}{\lambda+\lambda_j^k}(\dim \Gamma_j(X)-\dim \Gamma_j(Y))$$
$$=\lambda[\mathrm{Trace}\,(\lambda+(P^*P)^k)^{-1}-\mathrm{Trace}\,(\lambda+(PP^*)^k)^{-1}].$$

Since the right side is independent of λ, tending λ to infinity, we obtain the formula (1).

Now the asymptotic behaviour of $\mathrm{Trace}\,(\lambda+(P^*P)^k)^{-1}$ and $\mathrm{Trace}\,(\lambda+(PP^*)^k)^{-1}$ are known. See author's previous papers [4] and [5].

Let U be a coordinate patch of M where the bundles X and Y are trivial. We denote by $(x_1, x_2, \cdots, x_n)=x$ the coordinate of a point in U. Consider the $l\times l$ matrix valued function $a(x;\xi,\sigma)$ of x in U and of (ξ,σ) in $\boldsymbol{R}^{n+1}-\{0\}$ defined by

$$a(x;\xi,\sigma)=\sigma^{2mk}I+e^{-ix\cdot\xi}(P^*P)^k(e^{ix\cdot\xi}).$$

Next determine the formal series $b(x;\xi,\sigma)=\sum_{j=0}^{\infty} b_{-2mk-j}(x;\xi,\sigma)$ of $l\times l$ matrix valued functions b_{-2mk-j} homogeneous of degree $-2mk-j$ in ξ and σ by the generalized Leibniz formula

$$\sum_{|\alpha|\le 2mk}\frac{1}{\alpha!}D_\xi^\alpha a(x;\xi,\sigma)D_x^\alpha b(x;\xi,\sigma)=I \tag{2}$$

where I is the identity matrix. Then we have the asymptotic formula

$$\mathrm{Trace}\,(\sigma^{2mk}+(P^*P)^k)^{-1}$$
$$\sim\sum_{j=0}\sigma^{-2mk+n-j}(2\pi)^{-n}\int_M \frac{d\mu(x)}{\rho(x)}\int_{\boldsymbol{R}^n}\mathrm{trace}\; b_{-2mk-j}(x;\xi,1)d\xi.$$

Therefore the formula (1) gives the following equalities:

$$\begin{aligned}&\int_M \frac{d\mu(x)}{\rho(x)}\int_{\boldsymbol{R}^n}\mathrm{trace}\, b_{-2mk-j}(x;\xi,1)d\xi\\ &=\int_M \frac{d\mu(x)}{\rho(x)}\int_{\boldsymbol{R}^n}\mathrm{trace}\, b'_{-2mk-n}(x;\xi,1)d\xi,\end{aligned} \tag{3}$$

for $j=0,1,2,\cdots,n-1,$

and

$$\begin{aligned}&\mathrm{Ind}\,(P)=A(k)-A'(k),\\ &A(k)=(2\pi)^{-n}\int_M \frac{d\mu(x)}{\rho(x)}\int_{\boldsymbol{R}^n}\mathrm{trace}\, b_{-2mk-n}(x;\xi,1)d\xi,\\ &A'(k)=(2\pi)^{-n}\int_M \frac{d\mu(x)}{\rho(x)}\int_{\boldsymbol{R}^n}\mathrm{trace}\, b'_{-2mk-n}(x;\xi,1)d\xi,\end{aligned} \tag{4}$$

where b'_j are the functions formed from PP^* in just the same process as b_j are formed from P^*P and $\rho(x)$ is the density of the volume element $d\mu(x)$ on M.

It is possible to simplify the formula (4) further.

Theorem 2. *Formula* (4) *holds for* $k=1$.

Proof. Set $\square=(P^*P+1)^{k_0}$ with a sufficiently large fixed k_0. From the operator calculas we have, for $\lambda>2$ and $\mathrm{Re}\, s>0$,

$$(5)\qquad (\lambda^{2mk_0s}+\square^s)^{-1}=\frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta^s+\lambda^{2mk_0s}}(\zeta-\square)^{-1}d\zeta$$

where Γ is the complex contour from $-\infty i$ to ∞i along the imaginary axis and the branch of ζ^s is so taken that $1^s=1$. Thus, using the coordinate expression of X and Y, we have the following asymptotic expansion in (ξ,λ) as $|\xi|+|\lambda|\to\infty$. For any smooth function φ with compact support in U and for any constant vector v and real linear function $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$ of coordinate function $x_1,\cdots,x_n$,

$$(6)\qquad \begin{aligned} e^{-ix\cdot\xi}(\lambda^{2mk_0s}+\square^s)^{-1}\varphi e^{ix\cdot\xi}v &= \frac{1}{2\pi i}\int_\Gamma \frac{e^{-ix\cdot\xi}}{\lambda^{2mk_0s}+\zeta^s}(\zeta-\square)^{-1}(e^{ix\cdot\xi}\varphi v)d\zeta \\ &\sim \frac{1}{2\pi i}\sum_j\int_\Gamma \frac{1}{\lambda^{2mk_0s}+\zeta^s}b_{-2mk_0-j}(x;\xi,\zeta^{\frac{1}{2mk_0}})v\,d\zeta. \end{aligned}$$

Since $\int_\Gamma \frac{1}{\lambda^{2mk_0s}+\zeta^s}b_{-2mk_0-j}(x;\xi,\zeta^{\frac{1}{2mk_0}})$ is positively homogeneous in (ξ,λ) of degree $-2mk_0s-j$, $\mathrm{Trace}(\lambda^{2mk_0s}+\square^s)^{-1}$ has an asymptotic expansion in λ, that is,

$$\mathrm{Trace}\,(\lambda^{2mk_0s}+\square^s)^{-1} \sim\sum_j\frac{1}{2\pi i}\int_M\frac{d\mu(x)}{\rho(x)}\int_{R^n}d\xi\int_\Gamma\frac{1}{\lambda^{2mk_0s}+\zeta^s}\,\mathrm{trace}\, b_{-2mk_0-j}(x;\xi,\zeta^{\frac{1}{2mk_0}})d\zeta$$

(see [4] or [5]).

Therefore, if s is large enough,

$$(7)\quad A(k_0s)=\frac{\lambda^{2msk_0}}{2\pi i}\int_M\frac{d\mu(x)}{\rho(x)}\int_{R^n}d\xi\int_\Gamma\frac{1}{\zeta^s+\lambda^{2mk_0s}}\mathrm{trace}\, b_{-2mk_0-n}(x;\xi,\zeta^{\frac{1}{2mk_0}})d\zeta.$$

This is analytic in s, $\mathrm{Re}\, s>0$, and

$$(8)\qquad A(1)=\frac{\lambda^{2m}}{2\pi i}\int_M\frac{d\mu(x)}{\rho(x)}\int_{R^n}d\xi\int_\Gamma\frac{1}{\lambda^{2m}+\zeta^{\frac{1}{k_1}}}\,\mathrm{trace}\, b_{-2mk_0-n}(x;\xi,\zeta^{\frac{1}{2mk_0}})d\zeta.$$

On the other hand, (6) implies that the n-th term of the expansion of $e^{-ix\cdot\xi}(\lambda^{2m}+P^*P)^{-1}\varphi e^{ix\cdot\xi}v$ is equal to

$$\frac{1}{2\pi i}\int_\Gamma\frac{1}{\lambda^{2m}+\zeta^{\frac{1}{k_0}}}b_{-2mk_0-n}(x;\xi,\zeta^{\frac{1}{2mk_0}})d\zeta.$$

Thus this is equal to the n-th term calculated from the generalized Leibniz rule (2) where k is replaced by 1. This and (8) prove Theorem 2.

As a corollary to the formula (4) we shall give an analytic proof of

Theorem 3. ([2]). $\mathrm{Ind}\,(P)=0$, *if the dimension of M is odd.*

Proof. From the generalized Leibniz rule (2), the function b_{-n-2m} is odd in ξ. Therefore the integral

$$\int_{R^n}b_{-2m-n}(x;\xi,1)d\xi$$

vanishes.

References

[1] Atiyah, M. F.: Global aspects of the theory of elliptic differential operators. Abstract of reports, Congress at Moscow, 7-14 (1966).

[2] Atiyah, M. F., and Singer, I. M.: The index of elliptic operators on compact manifolds. Bull. Amer. Math. Soc., **72**, 422-433 (1963).

[3] Atiyah, M. F., and Bott, R.: A Lefschetz fixed point formula for elliptic complexes. I. Ann. Math., **68**, 374-407 (1967).

[4] Fujiwara, D.: The Asymptotic formula for the trace of the green operators of elliptic operators on compact manifolds. Proc. Japan Acad., **43**, 426-427 (1967).

[5] ——: On the asymptotic formula for the Green operators of elliptic operators on compact manifolds. J. Fac. Sci. Univ. Tokyo, **14**, 251-284 (1967).