# 32. On Generalized Integrals. I

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1. **Introduction.** Prof. K. Kunugi introduced, in 1954, the notion of ranked spaces, in the Note [2], as an extension of the metric space, and introduced further, in 1956, the notion of a generalized integral, in the Note [3], based on his theory of ranked spaces and called it the (E. R.) integral. In fact, to give the definition of his generalized integral, he started with the set $\mathcal{E}$ of step functions defined on a finite interval $a \leq x \leq b$, that is, functions having a constant value $\alpha_i$ in each of a finite number of sub-intervals $a_{i-1} < x < a_i$ in a division of $a \leq x \leq b$: $a_0 = a < a_1 \cdots < a_n = b$, as to the endpoints of these sub-intervals, we can assign values of the functions there arbitrarily. He supposed the integral defined for these functions, as usual, by the sum $\sum_i \alpha_i(a_i - a_{i-1})$. He introduced on the set $\mathcal{E}$ the set of neighbourhoods defined in the following way: Given a non-negative integer $\nu$, a closed subset $F$ of the interval $a \leq x \leq b$ and a point $f$ of $\mathcal{E}$, the neighbourhood $V(F, \nu; f)$ of $f$ is the set of all step functions $g(x)$ such that: $g(x) - f(x)$ is expressed as a sum of two step functions $p(x)$ and $\gamma(x)$ satisfying the following conditions:

[1] $\gamma(x) = 0$ for all $x \in F$,

[2] $\int_a^b |p(x)|\, dx < 2^{-\nu}$,

[3] $\left|\int_a^b \gamma(x) dx\right| < 2^{-\nu}$.

Under this topology, $\mathcal{E}$ becomes a uniform space the depth of which is $\omega_0$, and so the indicator of $\mathcal{E}$ should be $\omega_0$. The set $\mathfrak{B}_\nu$ of neighbourhoods of rank $\nu (\nu = 0, 1, 2, \cdots)$ is formed by the neighbourhoods $V(F, \nu; f)$ with mes $([a, b] \backslash F) < 2^{-\nu}$.[1] In this ranked space $\mathcal{E}$, we see that if $u: \{V(F_n, \nu_n; f_n)\}$ is a fundamental sequence of neighbourhoods, the limit $f(x) = \lim_{n \to \infty} f_n(x)$ exists almost everywhere, and the integrals $\int_a^b f_n(x) dx$ converges to a finit limit. This suggests that it should be possible to take this limit as the value of the integral of $f(x)$. As a formula

$$\int_a^b f(x) dx = \lim_{n \to \infty} \int_a^b f_n(x) dx.$$

To justify this convention, he showed that it does not depend on the particular choice of the fundamental sequence of neighbourhoods,

1) For the sets $E$ and $F$, $E \backslash F$ denotes the set of all those points of $E$ which do not belong to $E$.

that is, if $u: \{V_n(f_n)\}$ and $v: \{V_n(g_n)\}$ are two fundamental sequences belonging to the same maximal collection $u^*$ (he introduced the notion of maximal collection, corresponding to the notion of "equivalence" between fundamental sequences in a metric space), then we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x) \qquad \text{a.e.}$$

and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \lim_{n\to\infty} \int_a^b g_n(x)dx.$$

But, the set of functions integrable in this sense is not linear, nor the generalized integral is a linear functional. In order to guarantee both linearities, on the fundamental sequences he imposed additional conditions, precisely the conditions $P$ and $P^*$, which are independent on the notion of fundamental sequences.

The same reason will also be seen in the different definition from Kunugi's given by H. Okano in [5] for the (E. R.) integral.

On the other hand, I. Amemiya and T. Ando, in [1], proved that Kunugi's generalized integral[2] is identical with A-integral. Moreover, they showed that the set of functions integrable in the generalized sense is the completion of the set of Lebesgue integrable functions with respect to the quasi-norm

$$||f||_* = \sup_{k>0} \left| \int_a^b [f(x)]^k dx \right| + \sup_{k>0} k \operatorname{mes} \{x;\, |f(x)| \geq k\},$$

where the function $[f(x)]^k$, the truncation of $f(x)$ by the positive number $k$, is defined by

$$[f(x)]^k = \begin{cases} f(x) & \text{if } |f(x)| \leq k, \\ k \operatorname{sign} f(x) & \text{if } |f(x)| > k. \end{cases}$$

Hence, there arises a question whether, if we introduce suitably a set of neighbourhoods on a set $\mathcal{E}$ and define suitably a rank, the set of (E. R.) integrable functions can be obtained as a completion of the set $\mathcal{E}$. In papers I, II, and III, we will give a positive answer to it. We will construct a completion of $\mathcal{E}$, actually obtaining a ranked space of (E. R.) integrable functions to which the integral admits a unique and natural extension.

To do this, first we will show, in this paper, a few properties about the space $\mathcal{E}$.

Let us recall the notion of ranked spaces.

*Definition of the depth of the space.* Consider a set $R$ (called a space) endowed with such a structure that each point $p$ of $R$ has a non-empty set of subsets of $R$ (denoted by $V(p)$ and called neighbourhoods of $p$) satisfying the axioms (A) and (B) of Hausdorff.[3] Given a point $p$ of $R$, we say that a monotone decreasing sequence

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2) This integral, moreover, has been extended by K. Kunugi [4] and H. Okano [5].

3) F. Hausdorff: Grundzüge der Mengenlehre. Leipzig (Veit), p. 213 (1914).

of neighbourhoods $V_\alpha(p)$ is "*type* $\gamma$", where $\gamma$ is an ordinal number of Cantor, if $\alpha$ runs through the set $0 \leq \alpha < \gamma$ of all ordinal numbers and if $V_\alpha(p) \supseteq V_\beta(p)$ for all $\alpha, \beta$ with $0 \leq \alpha < \beta < \gamma$:

(1) $V_0(p) \supseteq V_1(p) \supseteq \cdots \supseteq V_\beta(p) \cdots, 0 \leq \alpha < \gamma.$

The sequence (1) which has no neighbourhoods $U(p)$ such that $\bigcap_\alpha V_\alpha(p) \supseteq U(p)$ is said to be "*maximal*". We denote by $\omega(R, p)$ the smallest ordinal number of types of maximal monotone decreasing sequences of neighbourhoods of $p$. Now, we consider such a space that there is at least one point having a maximal monotone decreasing sequence of neighbourhoods. The smallest ordinal number $\omega(R, p), p \in R$, is called the *depth of* $R$ and denoted by $\omega(R)$. $\omega(R)$ is an inaccessible ordinal number (or regular).[4]

*Definition of the ranked space.* Let us choose once for all an inaccessible ordinal number $\omega$ such that $\omega_0 \leq \omega \leq \omega(R)$. $\omega$ is called "indicator" of $R$. Given an ordinal number $\alpha$, which runs through the interval $0 \leq \alpha < \omega$, suppose that we have a set $\mathfrak{V}_\alpha$ of neighbourhoods, called *neighbourhoods of rank* $\alpha$. Then $R$ is said to be a ranked space, if the sequence of sets $\mathfrak{V}_\alpha (0 \leq \alpha < \omega)$ satisfies the following axiom (a):

(a) For every neighbourhood $V(p)$ of $p$ ($p \in R$) and for every ordinal number $\alpha$ such that $0 \leq \alpha < \omega$, there exist an ordinal number $\beta$ and a neighbourhood $U(p)$ of $p$ such that we have at the same time

$$\alpha \leq \beta < \omega, \; U(p) \subseteq V(p), \; U(p) \in \mathfrak{V}_\beta.$$

*Definition of the fundamental sequence or Cauchy sequence.* Let $\beta$ be an inaccessible ordinal number such that $0 < \beta \leq \omega$. Then a monotone decreasing sequence of neighbourhoods of points:

$$V_0(p_0) \supseteq V_1(p_1) \supseteq \cdots \supseteq V_\alpha(p_\alpha) \supseteq \cdots, 0 \leq \alpha < \beta,$$

is said to be *fundamental* or *of Cauchy*, if there is an ordinal number $\gamma(\alpha)$ such that $V_\alpha(p_\alpha) \in \mathfrak{V}_{\gamma(\alpha)}$ for all $\alpha$; $0 \leq \alpha < \beta$, and satisfies the following two conditions:

(1) $\gamma(0) \leq \gamma(1) \leq \cdots \leq \gamma(\alpha) \leq \cdots (0 \leq \alpha < \beta)$

(2) for each $\alpha$ such that $0 \leq \alpha < \beta$, there is a number $\lambda = \lambda(\alpha)$ such that $\alpha \leq \lambda < \beta, p_\lambda = p_{\lambda+1}$, and $\gamma(\lambda) < \gamma(\lambda+1)$(except the equality).

Given two monotone decreasing sequences of points $u: \{V_\alpha(p_\alpha)\}$ and $v: \{V_\beta(q_\beta)\}$, we denote by $u \succ v$ the relation between $u$ and $v$ such that for every $V_\alpha(p_\alpha)$ there is a $V_\beta(q_\beta)$ contained in $V_\alpha(p_\alpha)$. We will introduce, with Y. Yoshida, the notion of maximal collection in a slight different form from Kunugi's. A set $u^*$ of fundamental sequence

4) A limit number $\alpha$ is said to be "inaccessible", if, for every $\beta$ with $\beta < \alpha$ and for every function $\alpha(\gamma)$ defined for $\gamma$ with $0 \leq \gamma < \beta$, such that $0 \leq \alpha(\gamma) < \alpha$, we have always $\sup_\gamma \alpha(\gamma) < \alpha$.

is said to be a *maximal collection*, if it satisfies the following conditions:

(1*) for $u \in u^*$ and $v \in u^*$, there is a $w \in u^*$ such that $w \succ u$ and $w \succ v$,

(2*) there is no set $v^*$ of fundamental sequences with the property (1*), strictly containing $u^*$.

**2. Spaces of step functions.** To fix the ideas, we consider the real valued functions $y=f(x)$ defined on the interval $[a, b]$, that is, the set of all $x$ such that $a \leq x \leq b$, where $a, b$ are arbitrary two real numbers such that $a<b$. We start with the set $\mathcal{E}$ of step functions, following K. Kunugi, only to avoid assuming anything of the theory of integration, and we suppose the integral defined by the sum $\sum \alpha_i(a_i - a_{i-1})$ for the step function having a constant value $\alpha_i$ in each of a finite number of sub-intervals $a_{i-1}<x<a_i$.

Let us now introduce on the set $\mathcal{E}$ a set of neighbourhoods.

**Definition 1.** Given a closed subset $F$ of $[a, b]$, a positive number $\varepsilon$ and a point $f$ of $\mathcal{E}$, the neighbourhood of $f$, denoted by $V(F, \varepsilon; f)$ or simply by $V(f)$, is the set of all those step functions $g(x)$ which are the sums of $f(x)$ and the other functions $\gamma(x)$ having the following properties:

$[\alpha]$ $|r(x)|<\varepsilon$ for all $x \in F$

$[\beta]$ $k \operatorname{mes}\{x; |r(x)|>k\}<\varepsilon$ for each $k>0$,

$[\gamma]$ $\left|\int_a^b [r(x)]^k dx\right|<\varepsilon$ for each $k>0$,

where $[r(x)]^k$ is the truncation of $r(x)$ by $k$. Then, the neighbourhoods satisfy the axioms (A) and (B) of Hausdorff. To see the depth of $\mathcal{E}$, we first show the following lemma:

**Lemma 1.** *If the neighbourhoods $V(A, \varepsilon, f)$ and $V(B, \eta; g)$ have the relation $V(A, \varepsilon; f) \supseteq V(B, \eta; g)$, then* $\operatorname{mes}(A \backslash B)=0$ *and* $\varepsilon \geq \eta$.

**Proof.** Suppose, if possible, that $\operatorname{mes}(A \backslash B)>0$. Let $\eta'$ be a positive number with $0<\eta'<\eta$, then there is a finite set of disjoint intervals $I_i (i=1, 2, \cdots, i_0)$ contained in the set $[a, b] \backslash B$ and with $\eta'/2\varepsilon > \operatorname{mes}(\bigcup_i I_i) > \operatorname{mes}((\bigcup_i I_i) \cap A)>0$. Put $r(x)=\eta'/\operatorname{mes}(\bigcup_i I_i)$ on $\bigcup_i I_i$ and zero elsewhere, then $h(x)=g(x)+r(x) \in V(B, \eta; g)$, but $\notin V(A, \varepsilon; f)$. Next suppose that $\varepsilon<\eta$, if possible. Since $\operatorname{mes}(A \backslash B)=0$, there is a finite set of disjoint intervals $I_i (i=1, 2, \cdots, i_0)$ such that $\operatorname{mes}(A')>0$ and $\operatorname{mes}(\bigcup_i I_i)<1$, where $A'=(\bigcup_i I_i) \cap A \cap B$. Put $\alpha=\max_{x \in A'} |f(x)-g(x)|$ and let $\lambda$ be a number $\geq 1$ with $\operatorname{mes}\{x; |f(x)-g(x)| \geq \alpha/\lambda, x \in A\}>0$, then there is a $\alpha'$ with $\varepsilon<\alpha/\lambda+\alpha'<\eta$, since $\alpha/\lambda \leq \alpha<\varepsilon<\eta$. Put $r(x)=\alpha' \operatorname{sign}(f(x)-g(x))$ on $\bigcup I_i$ and zero elsewhere, then $h(x)=g(x)+r(x) \in V(B, \eta; g)$ but $\notin V(A, \varepsilon; f)$, in contradiction to the hypothesis.

**Proposition 1.** *$\mathcal{E}$ is a space the depth of which $\omega_0$.*

**Proof.** For each $f \in \mathcal{E}$, the monotone decreasing sequence of neighbourhoods $\{V([a, b], 1/n; f)\}$ is maximal. For, if not, there is a $V(B, \eta; f)$ with $V(B, \eta; f) \subseteq \bigcap_n V([a, b], 1/n; f)$, and then $\eta = 0$ from Lemma 1, contrary to $\eta > 0$. Hence $\omega(R) = \omega_0$.

For $\nu = 0, 1, 2, \cdots$, a neighbourhood $V(F, \nu; f)$ is said to be rank $\nu$, if it satisfies the condition

$[\delta]$ mes$([a, b] \backslash F) < \varepsilon$ and $\varepsilon = 2^{-\nu}$,

and by $\mathfrak{B}_\nu$ the set of all neighbourhoods of rank $\nu$. Then, the rank so defined satisfies the condition (a). For, given a neighbourhood $V(A, \varepsilon; f)$, we have $V([a, b], 1/n; f) \subseteq V(A, \varepsilon; f)$ for every $n$ with $1/2^n < \varepsilon$. Therefore, it follows that:

**Proposition 2.** *$\mathcal{E}$ is a ranked space.*

**Lemma 2.** *Let $u: \{V(A_n, \varepsilon_n; f_n)\}$ be a monotone decreasing sequence for which* mes$([a, b] \backslash A_n)$ *and $\varepsilon_n$ converge to zero as $n \to \infty$, then*

1) *$f_n(x)$ converges to a finite function $f(x)$ almost everywhere on $[a, b]$, precisely $\bigcup_{m=0}^{\infty} \left( \bigcap_{m=n}^{\infty} A_m \right)$.*

2) *the integrals $\int_a^b f_n(x)dx$ converges to a finite limit.*

**Proof.** Let $x \in \bigcap_{m=n}^{\infty} A_m$, then for $\varepsilon > 0$, there is an $n_0$ such that $\varepsilon_{n_0} < \varepsilon$ and $n_0 > n$, and we have $|f_m(x) - f_{m'}(x)| < \varepsilon_m < \varepsilon$ for every $m' > m > n_0$, since $f_{m'} \in V_m(f_m)$ and $x \in A_m$. By Lemma 1, mes$(A_n)$ $=$ mes$\left( \bigcap_{m=n}^{\infty} A_m \right)$, hence (1) follows. For every $m > n$, $\left| \int_a^b (f_m(x) - f_n(x))dx \right|$ $< \varepsilon_n$ results from $f_m \in V_n(f_n)$. This proves (2).

**Corollary 1.** *When $u: \{V_n(f_n)\}$ is a fundamental sequence, $f_n(x)$ converges to a finite function $f(x)$ a.e. and $\int_a^b f_n(x)dx$ converges to a finite limit.*

We denote, from now onwards, by $J(u)$ the limit function $f(x)$ and by $I(u)$ the limit value $\lim_{n \to \infty} \int_a^b f_n(x)dx$.

**Lemma 3.** *Let $u: \{V(A_n, \varepsilon_n; f_n)\}$ be a monotone decreasing sequence for which* mes$([a, b] \backslash A_n)$ *and $\varepsilon_n$ converge to zero as $n \to \infty$, and put $f(x) = \lim_{n \to \infty} f_n(x)$ a.e., then*

1) $|f(x) - f_n(x)| \leq \varepsilon_n$ *for all* $x \in \bigcap_{m=n}^{\infty} A_m$,

2) $k\,$mes$\{x; |f(x) - f_n(x)| > k\} \leq \varepsilon_n$ *for each* $k > 0$,

3) $\left| \int_a^b [f(x) - f_n(x)]^k dx \right| \leq \varepsilon_n$ *for each* $k > 0$.

**Proof.** 1) and 3) are easily seen from Lemma 2. As to 2), since, if we put

$$E_m=\{x;\ |f_n(x)-f_{m'}(x)|>k \quad \text{for all } m'\geq m\},$$
$$E=\bigcup_{m=0}^{\infty} E_m,\ A=\bigcup_{n=0}^{\infty}\left(\bigcap_{m=n}^{\infty} A_m\right),$$

then $A\cap E=\{x;\ |f(x)-f_n(x)|>k\}$ and $E_m\subseteq E_{m+1}$, we have $k$ mes $\{x;\ |f(x)-f_n(x)|>k\}=k$ mes $(E\cap A)=\lim_{m\to\infty} k$ mes $(E_m\cap A)\leq \overline{\lim}_{m\to\infty} k$ mes $\{x;\ |f_m(x)-f_n(x)|>k\}\leq \varepsilon_n$.

**Lemma 4.** *Let* $u:\{V(A_n, \varepsilon_n; f_n)\}$ *and* $v:\{V(B_n, \eta_n; g_n)\}$ *be two monotone decreasing sequences for which* mes $([a,b]\backslash A_n)$, mes $([a,b]\backslash B_n)$, $\varepsilon_n$ *and* $\eta_n$ *converge to zero as* $n\to\infty$ *and* $u\succ v$. *Put*

$$f(x)=\lim_{n\to\infty} f_n(x) \ \ a.e.,\quad g(x)=\lim_{n\to\infty} g_n(x) \ \ a.e..$$

*Then we have*

1) $f(x)=g(x)$ *a.e.*,

2) $\lim_{n\to\infty}\int_a^b f_n(x)dx=\lim_{n\to\infty}\int_a^b g_n(x)dx.$

**Proof.** Since $u\succ v$, for each $n$, there is an $m$ such that $V_n(f_n)\supseteq V_m(g_m)$, so that by Lemma 1, mes $(A_n\backslash B_m)=0$ and $|g_m(x)-f_n(x)|<\varepsilon_n$ for all $x\in A_n$. By Lemma 3, $|f(x)-f_n(x)|\leq\varepsilon_n$ for almost all $x\in A_n$ and $|g(x)-g_m(x)|\leq\eta_m$ for almost all $x\in B_m$. Therefore (1) follows. We have $\left|\int_a^b (f_n(x)-g_m(x))dx\right|<\varepsilon_n$, since $g_m\in V_n(f_n)$, and so we have 2) by Lemma 2.

**Corollary 2.** *When* $u$ *and* $v$ *are two fundamental sequences such that* $u\succ v$, *we have* $J(u)=J(v)$ *a.e. and* $I(u)=I(v)$.

## References

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