79. On Banach Function Spaces

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The theory of Riesz spaces (i.e. a normed vector lattice) plays an important role in the theory of normed function spaces. The theory have been developed by W. A. J. Luxemburg and A. C. Zaanen (see [1], [2]).

First I explain some terminologies (see [2]). Let X be a non-empty set and μ a non-negative, countable additive measure on X. We denote by $(X, \sum, \mu)$ a σ-finite measure space. Let M be the set of all real valued, μ-measurable functions on X, and M^+ the set of all non-negative functions of M. A function seminorm ρ is a mapping of M^+ into the real numbers and has the seminorm properties and $\rho(u)\leqslant\rho(v)$ if $u(x)\leqslant v(x)$ almost everywhere on X. We extend the domain of ρ to the whole M by defining $\rho(f)=\rho(|f|)$. The normed function space L_ρ is the set of $f\in M$ such that $\rho(f)<\infty$. We assume that there is at least one $f\in M$ such that $0<\rho(f)<\infty$. We introduce two function seminorms ρ_1, ρ_2 as follows

$$\rho_1(f)=\operatorname*{Sup}_{\rho(g)\leq 1}\left\{\int|fg|d\mu\right\},\qquad \rho_2(f)=\operatorname*{Sup}_{\rho_1(g)\leq 1}\left\{\int|fg|d\mu\right\}.$$

A measurable subset B of X is called *ρ-purely infinite*, if $\rho(\chi_c)=\infty$ for every $C(\subset B)$ of positive measure. ρ is called a *saturated function seminorm* if there is no ρ-purely infinite subsets. There is no loss of generality even if we remove the maximal ρ, ρ_1-purely infinite sets X_∞, X'_∞ from X (see Theorem 12. 1 in [2]). Then ρ, ρ_1, ρ_2 become the saturated function norms. We only use saturated function norms. Under this assumption, there is a sequence (π); $X_n\uparrow X$ such that $0<\mu(X_n)<\infty$ and $0<\rho(\chi_{X_n})<\infty$ (see Theorem 8.7 in [2]). We call such a sequence $(\pi): X_n\uparrow X$ a *ρ-exhaustive sequence*. We introduce the partial ordering in L_ρ by the following way: $f\leq g$ if and only if $f(x)\leq g(x)$ almost everywhere on X. Then L_ρ is a Riesz space with respect to the above ordering. Futher every nonempty subset of L_ρ which is bounded from above has a least upper bound in L_ρ, and it can be obtained by picking out an appropriate increasing subsequence. Such a Riesz space is called *super Dedekind complete*.

Let L_ρ^* be the Banach dual of L_ρ, and $L_{\rho,c}^*$ the subset of L_ρ^* having the following property; $F(\in L_\rho^*)$ belongs to $L_{\rho,c}^*$ if and only if $|f_n(x)|\downarrow 0$ (a.e) implies $F(|f_n|)\to 0$.

We shall now define two subsets of L_ρ as follows.

$L_\rho^a=\{f\in L_\rho : |f|\geqslant u_1\geqslant u_2\geqslant\cdots\downarrow 0 \text{ then } \rho(u_n)\to 0\}$.

L_ρ^π is the norm closure of the ideal generated by

$$\{\chi_{X_n} : X_n\in(\pi) : \rho\text{-exhaustive sequence}\}.$$

The latter definition has the meaning by the fact that ρ is a saturated function norm. To prove a fundamental theorem, we need

Lemma 1. $\rho_1(g)=\sup\left\{\int|fg|d\mu : f\in L_\rho^\pi,\ \rho(f)\leq 1\right\}$.

Proof. For α satisfying $\rho_1(g)>\alpha$, we can take $f_0\in L_\rho$ such that $\rho(f_0)\leq 1$ and $\int|f_0g|d\mu>\alpha$. For each n we put $f_n=\text{Min}\ (f_0, n\chi_{X_n})$ (where $X_n\in(\pi)$), then $\{f_n\}\subset L_\rho^\pi$ and $|f_ng|\uparrow|f_0g|$. There is a number n_0 such that $\int|f_mg|d\mu\geq\alpha$ for any $m\geq n_0$ and $\rho(f_m)\leq\rho(f_0)\leq 1$. Hence we have $\text{Sup}\left\{\int|fg|d\mu : f\in L_\rho^\pi,\ \rho(f)\leq 1\right\}>\alpha$. Therefore $\rho_1(g)\leq\text{Sup}\{\int|fg|d\mu : f\in L_\rho^\pi,\ \rho(f)\leq 1\}$. The another inequality is trivial.

Corollary 1. *The next statements are equivalent.*

(*i*) *A μ-measurable function f belongs to L_{ρ_1}.*

(*ii*) *$\int|fg|d\mu<\infty$ for every g in L_ρ^π, and $F(g)\equiv\int fgd\mu$ is a bounded linear functional on L_ρ^π.*

Proof. $L_\rho^\pi\subseteq L_\rho$ implies (i)$\to$(ii). Next we shall prove (ii)$\to$(i). For any g in L_ρ^π, we put $g_1=|g|/sgnf$. By the hypothesis $\int|fg|d\mu=\int fg_1d\mu=F(g_1)\leq\|F\|\rho(g)<\infty$ holds. Therefore by Lemma 1, we have $\rho_1(f)<\infty$, i.e. $f\in L_{\rho_1}$.

The next theorem was first proved by W. A. J. Luxemberg with the hypothesis that L_ρ is complete with respect to the function norm ρ, but without this hypothesis we can prove it.

Theorem 1. *In order that $G\in L_\rho^*$ belongs to L_{ρ_1}, it is necessary and sufficient that $G(f_n)$ tends to zero for every sequence $f_n\in L_\rho$ satisfying $f_n(x)\downarrow 0$ on X.*

Proof of necessity. For $G\in L_{\rho_1}$, there is a $g\in L_{\rho_1}$ such that $G(f_n)=\int f_ngd\mu$. If $f_n\downarrow 0$, then $G(f_n)\to 0$.

Proof of sufficiency. There is a ρ-exhaustive sequence $(\pi):X_n\uparrow X$. For any μ-measurable set $E\subset X_1$, we define a set function $F(E)$ by $F(E)=G(\chi_E)$. It is a countably additive, μ-absolutly continuous set function. Therefore by Radon-Nikodym Theorem, there is an integrable function $g(x)$ on X_1 such that $F(E)=G(\chi_E)=\int g\chi_Ed\mu$ for any measurable set $E\subset X_1$. By the same argument for $X_2, X_3, \cdots$ $G(\chi_E)=\int g\chi_Ed\mu$ holds for any measurable set E included in some X_n. For any step functions $f(x)$ on some $X_i\in(\pi)$, the same equality holds. For any

non-negative, bounded functions $f(x)$ whose support is contained in some X_n, $G(f)=\int fg d\mu$ holds.
Even if f is not non-negative, the same argument can be applied, and

$$G(f)=\int fg d\mu$$

holds for any $f \in L_\rho^\pi$.
Next we show that g is a member of L_{ρ_1}. $G(f)$ is bounded on L_ρ^π, hence $\int |fg| d\mu < \infty$ holds for any $f \in L_\rho^\pi$. Therefore by Corollary 1, $\rho_1(g) < \infty$, i.e. $g \in L_{\rho_1}$. Next we show that for any $f \in L_\rho$, $G(f)=\int fg d\mu$. There exists $\int fg d\mu$ for any $f \in L_\rho$. For simplicity we suppose that $f \geq 0$. For each n, we put $f_n = \mathrm{Min}(f, n\chi_{X_n})$. Since $f_n \in L_\rho^\pi$ we have $G(f_n) = \int f_n g d\mu$, and by the dominated convergence Theorem we have $\int f_n g d\mu \to \int fg d\mu$. Therefore $G(f_n) \to \int fg d\mu$. Since $f - f_n \in L_\rho$ and $f - f_n \downarrow 0$, we have $G(f-f_n) \to 0$ by the hypothesis. Therefore $G(f)=\int fg d\mu$.

Corollary 2. $L_{\rho,c}^* = L_{\rho_1}$.

Proof. If $G(\in L_\rho^*)$ belongs to L_{ρ_1}, then from Theorem 1 $G(f_n)\to 0$ for any $f_n \in L_\rho$ such that $f_n \downarrow 0$ (a.e.). Therefore we have $G \in L_{\rho,c}^*$. Next if $G \in L_\rho^*$ belongs to $L_{\rho,c}^*$, then by the definition of $L_{\rho,c}^*$, we have $G(f_n)\to 0$ for any $f_n \in L_\rho$ satisfying $f_n \downarrow 0$ (a.e.). It follows from Theorem 1 that $G \in L_{\rho_1}$. Therefore we have the desired results $L_{\rho,c}^* = L_{\rho_1}$.

If ρ is a function norm, $\{f \in L_\rho : G(f)=0$ for any $G \in L_{\rho,c}^*\} = \{0\}$ holds (see Theorem 15.2 in [2]). Therefore from Corollary 2 $\{f \in L_\rho : G(f)=0$ for any $G \in L_{\rho,c}^*\} = \{0\}$ holds. Furthere ρ is a saturated function norm, the sequence $\{\chi_{X_n} : X_n \in (\pi)\}$ is a countable basis of L_ρ. By the above result, we have two Theorems obtained by W. A. J. Luxemberg and A. C. Zaanen (see Theorem 25, 10, and Corollary 24.3 in [2]).

Theorem A. *The next conditions are equivalent.*

(i) *L_ρ^a is order dense in L_ρ (i.e. the ideal generated by L_ρ^a coincides with L_ρ).*

(ii) *$(L_\rho^a)^* = L_{\rho,c}^*$ (algebraically and isometrically).*

(iii) *$L_\rho^a = L_\rho^\pi$ for at least one ρ-exhaustive sequence (π).*

Theorem B. *$L_\rho^* = L_{\rho,c}^*$ if and only if $L_\rho = L_\rho^a$.*

Corollary 3. *$L_\rho^* = L_{\rho_1}$ if and only if $L_\rho = L_\rho^a$.*

Proof. By Corollary 1 and Theorem B, Corollary 3 follows.

Theorem 2. (*i*) *Suppose that $u_n \uparrow u$ (a.e.) and* $\lim \rho(u_n) < \infty$ *implies $\rho(u) < \infty$. $L_\rho = L_\rho^{**}$ (algebraically) holds if and only if $L_\rho = L_\rho^a$ and $L_{\rho_1} = L_{\rho_1}^a$.*

(*ii*) *Suppose* $u_n \uparrow u$ *(a.e.) implies* $\rho(u_n) \uparrow \rho(u)$. *(i.e. Fatou property) Then* $L_\rho = L_\rho^{**}$ *(algebraically and isometrically) holds if and only if* $L_\rho = L_\rho^a$ *and* $L_{\rho_1} = L_{\rho_1}^a$.

Proof. (i) From the assumption, it follows that $L_\rho = L_{\rho_2}$ (algebraically) (see Theorem 7.7 in [2]).

Necessity. Suppose $0 \leqslant u_n \downarrow 0$ in $L_\rho = L_\rho^{**}$, then for any G in L_ρ^*, we have $G(u_n) = u_n(G) \to 0$. By the difinition of $L_{\rho,c}^*$, it follows that $L_\rho^* = L_{\rho,c}^*$ and $L_\rho = L_\rho^a$ by Corollay 3. By the same way if $0 \leqslant u_n \downarrow 0$ in L_ρ^*, then for any f in $L_\rho^{**} = L_\rho$, it follows $f(u_n) = u_n(f) \to 0$. We have $L_{\rho_1}^* = (L_{\rho_1})_c^*$ and $L_{\rho_1} = L_\rho$, and consequently $(L_\rho^*)^* = (L_{\rho,c}^*)^* = (L_\rho^*)_c^* = (L_{\rho,c}^*)_c^*$. Therefore we have $L_{\rho_1}^* = (L_{\rho_1})_c^*$ and $L_{\rho_1} = L_{\rho_1}^a$.

Sufficiency. If $L_\rho = L_\rho^a$, then $L_\rho^* = L_{\rho_1}$ holds from Corollary 3. By the same way above we have $L_{\rho_1}^* = L_{\rho_2}$ from $L_{\rho_1} = L_{\rho_1}^a$, therefore we have $L_\rho^{**} = (L_{\rho_1})^* = L_{\rho_2}$. And $L_\rho = L_{\rho_2}$ (algebraically) holds, the first assurtion was obtained.

(ii) $\rho = \rho_2$ holds if and only if ρ satisfies the hypothesis. Therefore we have $L_\rho^{**} = L_\rho$ algebraically and isometrically. This completes the proof of (ii).

Lemma 2. *If* $0 \leqslant u_n \downarrow$ *(a.e.) in* L_ρ, *and* $\varphi(u_n) \to 0$ *for every* φ *in* L_ρ^* *(the sequence converges weakly to zero), then* $\rho(u_n) \downarrow 0$ *(the sequence converges strongly to zero).*

Proof. By the difinition of $L_{\rho;c}^*$, we have $L_\rho^* = L_{\rho,c}^*$. Then by Theorem B, we have $L_\rho = L_\rho^a$ and $\rho(u_n) \downarrow 0$.

Theorem 3. *If* L_ρ^a *is order dense in* L_ρ, *we have* $\rho = \rho_2$ *on* L_ρ^a.

Proof. Since L_ρ^a is order dense in L_ρ, we have $L_{\rho,c}^* = (L_\rho^a)^*$ by Theorem A. If $f_n \downarrow 0$ (a.e.) in L_ρ^a, it follows that $\varphi(f_n) \to 0$ for any φ in $(L_\rho^a)^* = L_{\rho,c}^*$. By Lemma 2, $\{f_n\}$ converges to zero with the norm on L_ρ^a.

Let the sequence $\{g_n\}$ be $g_n \uparrow g$ (a.e.) in L_ρ^a, $0 \leqslant g - g_n \downarrow 0$ in L_ρ^a. Then $\rho(g - g_n)$ converges to zero, and $\rho(g) \leq \rho(g_n) + \rho(g - g_n)$, we have $\rho(g_n) \uparrow \rho(g)$. Therefore ρ has the Fatou property, and we obtain $\rho = \rho_2$ on L_ρ^a.

Corollary 4. *If* $L_\rho^a = L_\rho^\pi$ *(or equivalently* L_ρ^a *is order dense in* L_ρ*),* $P^+ = \{u \in L_\rho : \rho(u) = \rho_2(u)\}$ *is order dense.*

Proof. By Theorem 3, P^+ coincides with L_ρ^a.

References

[1] W. A. J. Luxemberg: Banach function spaces. Thesis Delft Inst. Techn., Assen (Netherland) (1955).

[2] W. A. J. Luxemberg and A. C. Zaanen: Notes on Banach function spaces. Proc. Acad. Sci., Amsterdam Note I, II, A 66, 135–153 (1963), Note III, IV, A 66, 239–263 (1963), Note V, A 66, 496–504 (1963), Note VI, VII, A 66, 655–681 (1963).