75. *On the Standard Complexes of Cotriples*

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In this paper we show that the standard complexes of cotriples are acyclic in most cases. This generalizes Proposition 4.1 of Eilenberg and Moore [4]. For example we confirm that the ordinary (co)homology of modules over an arbitrary ring (Cartan and Eilenberg [2] Chap. VI) is a cotriple (co)homology.

We follow the notions and notations of Eilenberg and Moore [3], [4]. Let $\mathcal{A}$ be a pre-additive category with kernels, $\boldsymbol{G}=(G, \varepsilon, \delta)$ be a cotriple in $\mathcal{A}$. Then $\{G^{n+1}\}_{n\geqq 0}$ is a simplicial object with the face operators $\varepsilon^i=G^i\varepsilon G^{n-i}: G^{n+1}\longrightarrow G^n$ and the degeneracy operators $\delta^i=G^i\delta G^{n-i}: G^{n+1}\longrightarrow G^{n+2}$. The *standard complex* of the cotriple $\boldsymbol{G}$ is defined by the sequence

$$(1)\qquad \cdots\longrightarrow G^{n+1}\xrightarrow{\partial_n} G^n\longrightarrow\cdots\xrightarrow{\partial_1} G\xrightarrow{\varepsilon} 1_{\mathcal{A}}\longrightarrow 0$$

where $\partial_n=\sum_{i=0}^n(-1)^i\varepsilon^i$. Let $\mathcal{G}$ be a projective class of sequences in $\mathcal{A}$ such that $\mathcal{G}$-projective objects are the objects $G(A)$, $A\in ob\mathcal{A}$ and their retracts.

Theorem. *In the above situation, the sequence*

$$(2)\qquad \cdots\longrightarrow G^{n+1}(A)\xrightarrow{\partial_n} G^n(A)\longrightarrow\cdots\xrightarrow{\partial_1} G(A)\xrightarrow{\varepsilon} A\longrightarrow 0$$

is a $\mathcal{G}$-projective resolution of A for any object A in $\mathcal{A}$.

Proof. Every cotriple is generated by an adjoint pair of functors ([4] Theorem 2.2 or [5]):

$$(3)\qquad (\varepsilon, \eta): S\dashv T: (\mathcal{A}, \mathcal{B})$$

i.e. $\boldsymbol{G}=(G, \varepsilon, \delta)$ is represented by $(ST, \varepsilon, S\eta T)$. We may suppose that $\mathcal{B}$, S, T are pointed. Since $\varepsilon S\cdot S\eta=1_S$, an object $A\in ob\mathcal{A}$ is a retract of $G(A')=ST(A')$ for some $A'\in ob\mathcal{A}$ if and only if A is a retract of $S(B)$ for some $B\in ob\mathcal{B}$. Hence the isomorphism of functors

$$(4)\qquad \mathcal{A}(S,\ \ )\longrightarrow\mathcal{B}(\ \ ,T)$$

implies $\mathcal{G}=T^{-1}\mathcal{E}_0$ where $\mathcal{E}_0$ is the class of all split exact sequences in $\mathcal{B}$. To prove the theorem we may show that the sequence

$$(5)\qquad \cdots\longrightarrow TG^{n+1}(A)\xrightarrow{T\partial_n} TG^n(A)\longrightarrow\cdots\xrightarrow{T\partial_1} TG(A)\xrightarrow{T\varepsilon} T(A)\longrightarrow 0$$

is split exact for every $A\in ob\mathcal{G}$.

Define morphisms $t_n: G^m\longrightarrow G^m$ and $u_n: G^m\longrightarrow G^{m+1}$, $n<m$, as follows

$$t_n=(1-\delta^0\varepsilon^1)(1-\delta^1\varepsilon^2)\cdots(1-\delta^{n-1}\varepsilon^n),$$

$$u_n = t_0\delta^0 - t_1\delta^1 + \cdots + (-1)^{n-1}t_{n-1}\delta^{n-1}$$

for $n \geq 1$ and $t_0 = 1$, $u_{-1} = 0$, $u_0 = 0$.

Then we have

$$\varepsilon^0 t_n = t_{n-1}\partial_n,\ \varepsilon^i t_n = 0, \quad 0 < i \leq n, \tag{6}$$

$$t_n + \partial_{n+1}u_n + u_{n-1}\partial_n = 1, \quad n \geq 0. \tag{7}$$

The relation (7) follows from (6) and the calculation:

$$\begin{aligned} & t_n + \partial_{n+1}u_n + u_{n-1}\partial_n \\ &= t_n + \partial_n(u_{n-1} + (-1)^{n-1}t_{n-1}\delta^{n-1}) + (-1)^{n+1}\varepsilon^{n+1}u_n \\ &\quad + (u_{n-2} + (-1)^{n-2}t_{n-2}\delta^{n-2})\partial_{n-1} + (-1)^n u_{n-1}\varepsilon^n \\ &= (t_n + (-1)^{n+1}\varepsilon^{n+1}u_n + (-1)^n u_{n-1}\varepsilon^n) + \partial_n u_{n-1} + u_{n-2}\partial_{n-1} \\ &\quad + (-1)^{n-1}\partial_n t_{n-1}\delta^{n-1} + (-1)^n t_{n-2}\delta^{n-2}\partial_{n-1} \\ &= t_{n-1} + \partial_n u_{n-1} + u_{n-2}\partial_{n-1}. \end{aligned}$$

By the isomorphism (3), $T(A)$ becomes an abelian group object in $\mathscr{B}$ for every $A \in ob\mathscr{A}$. We have $Tf\cdot(g+h) = Tf\cdot g + Tf\cdot h$ for $f \in \mathscr{A}(A', A)$, $g, h \in \mathscr{B}(B, T(A'))$, and $T(f+g) = Tf + Tg$ for $f, g \in \mathscr{A}(A', A)$, and $(g+h)\cdot k = g\cdot k + h\cdot k$ for $g, h \in \mathscr{B}(T(A'), T(A))$, $k \in \mathscr{B}(B, T(A'))$. Thus we can define

$$\begin{aligned} & s_n = \eta TG^{n+1}\cdot Tt_n + Tu_n : TG^{n+1} \longrightarrow TG^{n+2}, \quad n \geq 0, \\ & s_{-1} = \eta T : T \longrightarrow TG. \end{aligned}$$

Then (6) and (7) imply

$$\begin{aligned} & T\partial_{n+1}\cdot s_n + s_{n-1}\cdot T\partial_n \\ &= T\partial_{n+1}\cdot(\eta TG^{n+1}\cdot Tt_n + Tu_n) + (\eta TG^n\cdot Tt_{n-1} + Tu_{n-1})\cdot T\partial_n \\ &= Tt_n - \eta TG^n\cdot T(\varepsilon^0 t_n) + T(\partial_{n+1}u_n) + \eta TG^n\cdot T(t_{n-1}\partial_n) + T(u_{n-1}\partial_n) \\ &= 1_{TG^{n+1}}, \quad n \geq 1, \end{aligned}$$

and $s_{-1}\cdot T\varepsilon = 1_T$. Therefore the sequence (5) is split exact for every $A \in ob\ \mathscr{A}$. The proof is completed.

Remark. Since $t_n^2 = t_n$ for $n \geq 0$ by (6), G^{n+1} is a biproduct of the kernels of t_n and $1-t_n$. The kernel functor of $1-t_n$

$$0 \longrightarrow \tilde{G}_n \xrightarrow{\iota_n} G^{n+1} \xrightarrow{1-t_n} G^{n+1}$$

leads to

$$\tilde{G}_n = \cap_{i=1}^n \operatorname{Ker} \varepsilon^i$$

i.e. the sequence of abelian groups

$$0 \longrightarrow \mathscr{A}(A', \tilde{G}(A)) \xrightarrow{\mathscr{A}(\ ,\iota_n)} \mathscr{A}(A', G^{n+1}(A)) \xrightarrow{(\mathscr{A}(\ ,\varepsilon^i))} \Pi_{i=1}^n \mathscr{A}(A', G^{n+1}(A))$$

is split exact for every $A', A \in ob\ \mathscr{A}$, where the last homomorphism assigns to f an element $(\varepsilon^1(A)\cdot f, \varepsilon^2(A)\cdot f, \cdots, \varepsilon^n(A)\cdot f)$. Two morphisms of functors ε^0 and $\partial_n : G^{n+1} \longrightarrow G^n$ induce the same morphism $\tilde{\partial}_n : \tilde{G}_n \longrightarrow \tilde{G}_{n-1}$. $\{\tilde{G}_n, \tilde{\partial}_n\}_{n\geq 0}$ is the Moore complex of the simplicial object $\{G^{n+1}\}_{n\geq 0}$. The canonical resolution (see [4]) is a retract of the Moore complex.

Example. Let K be a ring with unit. Let ${}_K\mathscr{M}$, $\mathcal{S}$ be the categories of left K-modules and pointed sets respectively. Let $U : {}_K\mathscr{M}$

$\longrightarrow \mathcal{S}$ be the forgetful functor. Let $F: \mathcal{S} \longrightarrow {}_K\mathcal{M}$ be a functor such that, for every pointed set X, $F(X)$ is a free left K-module generated by the underlying set of $X-\{\text{the base point}\}$. Then we have an adjoint pair of functors $F \dashv U: ({}_K\mathcal{M}, \mathcal{S})$. Let $\boldsymbol{G}=(G, \varepsilon, \delta)$ be a cotriple generated by the pair $F \dashv U$. Then our theorem is available. Since $\varepsilon(A)$ is an epimorphism for every $A \in ob_K\mathcal{M}$, the class $\mathcal{G}=U^{-1}\mathcal{E}_0$ is exact. The sequence (2) is a free acyclic resolution for every K-module A. If $\hat{G}$ denotes the standard complex (1), then we have

$$\operatorname{Ext}{}^*_K(A, B)=H^*(\operatorname{Hom}{}_K(\hat{G}(A), B))$$

$$\operatorname{Tor}{}^K_*(C, A)=H_*(C\otimes_K\hat{G}(A))$$

for left K-modules A, B, and a right K-module C.

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