70. Calculus in Ranked Vector Spaces. III

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(1.7.6) It is obvious that one has the following; if $\{x_n\}$ is a quasi-bounded sequence and $\{a_n\}$ is a bounded sequence in $\mathfrak{R}$ (i.e., $|a_n|<M$, for $n=0, 1, 2, \cdots$), then $\{a_n x_n\}$ is also a quasi-bounded sequence.

In fact, let $\{\mu_n\}$ be a sequence in $\mathfrak{R}$ with $\mu_n \to 0$, then

$$\mu_n(a_n x_n)=(\mu_n a_n)x_n.$$

Since $|a_n|<M$, $\mu_n a_n \to 0$ in $\mathfrak{R}$. Using that $\{x_n\}$ is a quasi-bounded sequence, we have

$$\therefore \quad \{\lim \mu_n(a_n x_n)\} \ni 0.$$

(1.7.7) **Proposition.** *Let $l: E_1 \to E_2$ be a linear and continuous map between ranked vector spaces E_1, E_2. If $\{x_n\}$ is a quasi-bounded sequence in a ranked vector space E_1, then $\{l(x_n)\}$ is also a quasi-bounded sequence in E_2.*

Proof. Let $\{\mu_n\}$ be a sequence in $\mathfrak{R}$ such that $\mu_n \to 0$. Then it follows from the linearity of l that $\mu_n l(x_n)=l(\mu_n x_n)$. Using the assumption that $l: E_1 \to E_2$ is continuous,

$$\{\lim \mu_n l(x_n)\} \ni l(0)=0.$$

Therefore $\{l(x_n)\}$ is a quasi-bounded sequence.

(1.7.8) **Proposition.** *Let $E_1, E_2, \cdots, E_m$ be a family of ranked vector spaces. For a sequence $\{z_n\}=\{(x_{n1}, x_{n2}, \cdots, x_{nm})\}$ of the direct product $\times E_i$ to be a quasi-bounded sequence it is necessary and sufficient that, for each i $(i=1, 2, \cdots, m)$, $\{x_{ni}\}$ is a quasi-bounded sequence in E_i.*

Proof. Let $\{\mu_n\}$ be a sequence in $\mathfrak{R}$ with $\mu_n \to 0$. Then

$$\mu_n z_n=(\mu_n x_{n1}, \mu_n x_{n2}, \cdots, \mu_n x_{nm}).$$

By (1.5.1), $\{\lim \mu_n z_n\} \ni 0$ is equivalent to

$$\{\lim \mu_n x_{n1}\} \ni 0, \{\lim \mu_n x_{n2}\} \ni 0, \cdots, \{\lim \mu_n x_{nm}\} \ni 0.$$

That is, our assertion holds.

1.8. L-convergence. Let us introduce a new convergence in a ranked vector space E, where the convergence in the sense of (1.2.1) is defined.

(1.8.1) **Definition.** Let $\{x_n\}$ be a sequence of a ranked vector space E. We say that *a sequence $\{x_n\}$ converges to x in the sense of L-convergence*, and we write $\{\mathrm{Lim}\, x_n\} \ni x$ if and only if x_n can be written in the following way:

$$x_n - x = \lambda_n x'_n, \qquad n=0, 1, 2, \cdots$$

where $\{\lambda_n\}$ is a sequence in $\mathfrak{R}$ such that $\lambda_n \to 0$ and $\{x'_n\}$ is a quasi-bounded sequence in E.

Obviously $\quad \{\mathrm{Lim}\, x_n\} \ni x \Longleftrightarrow \{\mathrm{Lim}\,(x_n - x)\} \ni 0.$

(1.8.2) In particular, if $x_n = x$ for $n=0, 1, 2, \cdots$, then

$$\{\mathrm{Lim}\, x_n\} \ni x.$$

In fact, $x_n - x = 0 = \lambda_n 0$, where $\{\lambda_n\}$ is any sequence in $\mathfrak{R}$ with $\lambda_n \to 0$.

It is obvious that one has the following proposition:

(1.8.3) **Proposition.** *Let $\{x_n\}$ be a sequence of a ranked vector space E. Then*

$$\{\mathrm{Lim}\, x_n\} \ni x \quad \textit{implies} \quad \{\mathrm{lim}\, x_n\} \ni x.$$

(1.8.4) **Proposition.** *Let E be a ranked vector space, $\{x_n\}$, $\{y_n\}$ two sequences in E and $x, y \in E$. If $\{\mathrm{Lim}\, x_n\} \ni x$ and $\{\mathrm{Lim}\, y_n\} \ni y$ in E, then*

$$\{\mathrm{Lim}\,(x_n + y_n)\} \ni x + y.$$

Proof. If follows from definition of L-convergence that

$$x_n - x = \lambda_n x'_n, \; y_n - y = \mu_n y'_n, \quad \text{for } n=0, 1, 2, \cdots$$

where $\{\lambda_n\}$, $\{\mu_n\}$ are sequences in $\mathfrak{R}$ such that $\lambda_n \to 0$, $\mu_n \to 0$, and $\{x'_n\}$, $\{y'_n\}$ are quasi-bounded sequences in E.

$$\therefore \quad x_n + y_n - (x+y) = \lambda_n x'_n + \mu_n y'_n$$
$$= \tau_n \left\{ \frac{\lambda_n}{\tau_n} x'_n + \frac{\mu_n}{\tau_n} y'_n \right\}$$

where $\tau_n = \max(|\lambda_n|, |\mu_n|)$. Then

$$\left| \frac{\lambda_n}{\tau_n} \right| \le 1, \; \left| \frac{\mu_n}{\tau_n} \right| \le 1, \qquad \text{and} \quad \lim_{n\to\infty} \tau_n = 0.$$

By (1.7.5), (1.7.6)

$$\left\{ \frac{\lambda_n}{\tau_n} x'_n + \frac{\mu_n}{\tau_n} y'_n \right\}$$

is a quasi-bounded sequence. Hence

$$\{\mathrm{Lim}\,(x_n + y_n)\} \ni x + y.$$

(1.8.5) **Proposition.** *Let E be a ranked vector space, $\{x_n\}$ a sequence in E and $x \in E$. If $\{\mathrm{Lim}\, x_n\} \ni x$, then for any $\lambda \in \mathfrak{R}$*

$$\{\mathrm{Lim}\, \lambda x_n\} \ni \lambda x.$$

Proof. By assumption we have

$$x_n - x = \lambda_n x'_n, \qquad n=0, 1, 2, \cdots$$

where $\{\lambda_n\}$ is a sequence in $\mathfrak{R}$ with $\lambda_n \to 0$ and $\{x'_n\}$ is a quasi-bounded sequence in E.

$$\therefore \quad \lambda x_n - \lambda x = \lambda \lambda_n x'_n = (\lambda \lambda_n) x'_n$$
$$\therefore \quad \{\mathrm{Lim}\, \lambda x_n\} \ni \lambda x.$$

(1.8.6) **Proposition.** *Let E be a ranked vector space. If* $\lim \lambda_n = \lambda$ *in* $\mathfrak{R}$, *then for any* $x \in E$

$$\{\mathrm{Lim}\, \lambda_n x\} \ni \lambda x.$$

Proof. $$\lambda_n x - \lambda x = (\lambda_n - \lambda)x.$$

By assumption we have

$$\lambda_n - \lambda \to 0 \qquad \text{for} \quad n \to \infty$$

and if $x_n = x$ for $n = 0, 1, 2, \cdots$, by (1.7.3) $\{x_n\}$ is a quasi-bounded sequence

$$\therefore \quad \{\mathrm{Lim}\, \lambda_n x\} \ni \lambda x.$$

(1.8.7) **Proposition.** *Let E be a ranked vector space, $\{x_n\}$ a sequence in E, $\{\lambda_n\}$ a sequence in $\mathfrak{R}$, $x \in E$, and $\lambda \in \mathfrak{R}$. If* $\lim \lambda_n = \lambda$ *in* $\mathfrak{R}$ *and* $\{\mathrm{Lim}\, x_n\} \ni x$ *in E, then*

$$\{\mathrm{Lim}\, \lambda_n x_n\} \ni \lambda x.$$

Proof. (a) We shall show that our assertion holds in the following special case:

$$\lambda = 0, \qquad x = 0.$$

By assumption we have

$$x_n = \mu_n x'_n, \qquad n = 0, 1, 2, \cdots$$

where $\mu_n \to 0$ in $\mathfrak{R}$ and $\{x'_n\}$ is a quasi-bounded sequence in E.

$$\therefore \quad \lambda_n x_n = \lambda_n(\mu_n x'_n) = (\lambda_n \mu_n) x'_n$$

$$\therefore \quad \{\mathrm{Lim}\, \lambda_n x_n\} \ni 0.$$

(b) Let $\lim \lambda_n = \lambda$ and $\{\mathrm{Lim}\, x_n\} \ni x$, then

$$\lim (\lambda_n - \lambda) = 0 \quad \text{and} \quad \{\mathrm{Lim}\, (x_n - x)\} \ni 0.$$

By (a) we have

$$\{\mathrm{Lim}\, (\lambda_n - \lambda)(x_n - x)\} \ni 0$$

$$\therefore \quad \{\mathrm{Lim}\, (\lambda_n x_n - \lambda x_n - \lambda_n x + \lambda x)\} \ni 0.$$

By (1.8.2), (1.8.5), (1.8.6), we have

$$\{\mathrm{Lim}\, \lambda x_n\} \ni \lambda x, \ \{\mathrm{Lim}\, \lambda_n x\} \ni \lambda x, \quad \text{and} \quad \{\mathrm{Lim}\, (-\lambda x)\} \ni -\lambda x.$$

It follows from (1.8.4) that

$$\{\mathrm{Lim}\, \lambda_n x_n\} \ni \lambda x.$$

(1.8.8) **Proposition.** *Let $E_1, E_2, \cdots, E_m$ be a family of ranked vector spaces, $\{z_n\} = \{(x_{n1}, x_{n2}, \cdots, x_{nm})\}$ a sequence in the direct product $\times E_i$ and $z = (x_1, x_2, \cdots, x_m) \in \times E_i$. Then* $\{\mathrm{Lim}\, z_n\} \ni z$ *is equivalent to*

$$\{\mathrm{Lim}\, x_{n1}\} \ni x_1, \{\mathrm{Lim}\, x_{n2}\} \ni x_2, \cdots, \{\mathrm{Lim}\, x_{nm}\} \ni x_m.$$

Proof. (a) By assumption we have

$$z_n - z = \lambda_n z'_n, \qquad n = 0, 1, 2, \cdots$$

where $\lambda_n \to 0$ in $\mathfrak{R}$ and $\{z'_n\} = \{(x'_{n1}, x'_{n2}, \cdots, x'_{nm})\}$ is a quasi-bounded sequence in $\times E_i$. Then

$$x_{n1} - x_1 = \lambda_n x'_{n1},$$
$$x_{n2} - x_2 = \lambda_n x'_{n2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_{nm}-x_m=\lambda_n x'_{nm}.$$

Thus we have

$$\{\mathrm{Lim}\, x_{n1}\}\ni x_1, \{\mathrm{Lim}\, x_{n2}\}\ni x_2, \cdots, \{\mathrm{Lim}\, x_{nm}\}\ni x_m.$$

(b) Suppose conversely that

$$\{\mathrm{Lim}\, x_{n1}\}\ni x_1, \{\mathrm{Lim}\, x_{n2}\}\ni x_2, \cdots, \{\mathrm{Lim}\, x_{nm}\}\ni x_m.$$

It follows from definition of L-convergence that

$$\begin{aligned} x_{n1}-x_1&=\lambda_{n1}x'_{n1}, \\ x_{n2}-x\ &=\lambda_{n2}x'_{n2}, \\ &\cdots\cdots \\ x_{nm}-x_m&=\lambda_{nm}x'_{nm}, \end{aligned}$$

where $\lambda_{n1}\to 0$, $\lambda_{n2}\to 0, \cdots, \lambda_{nm}\to 0$ in $\mathfrak{R}$ and $\{x'_{n1}\}, \{x'_{n2}\}, \cdots, \{x'_{nm}\}$ are quasi-bounded sequences

$$\begin{aligned} \therefore\quad z_n-z&=(\lambda_{n1}x'_{n1}, \lambda_{n2}x'_{n2}, \cdots, \lambda_{nm}x'_{nm}) \\ &=\tau_n\left(\frac{\lambda_{n1}}{\tau_n}x'_{n1}, \frac{\lambda_{n2}}{\tau_n}x'_{n2}, \cdots, \frac{\lambda_{nm}}{\tau_n}x'_{nm}\right) \end{aligned}$$

where $\tau_n=\max(|\lambda_{n1}|, |\lambda_{n2}|, \cdots, |\lambda_{nm}|)$.

Then

$$\left|\frac{\lambda_{n1}}{\tau_n}\right|\leq 1, \left|\frac{\lambda_{n2}}{\tau_n}\right|\leq 1, \cdots, \left|\frac{\lambda_{nm}}{\tau_n}\right|\leq 1,$$

and

$$\tau_n\to 0.$$

By (1.7.6)

$$\left\{\frac{\lambda_{n1}}{\tau_n}x'_{n1}\right\}, \left\{\frac{\lambda_{n2}}{\tau_n}x'_{n2}\right\}, \cdots, \left\{\frac{\lambda_{nm}}{\tau_n}x'_{nm}\right\}$$

are quasi-bounded sequences, and so by (1.7.8)

$$\left\{\left(\frac{\lambda_{n1}}{\tau_n}x'_{n1}, \frac{\lambda_{n2}}{\tau_n}x'_{n2}, \cdots, \frac{\lambda_{nm}}{\tau_m}x'_{nm}\right)\right\}$$

is also a quasi-bounded sequence in $\times E_i$.

$$\therefore\quad \{\mathrm{Lim}\, z_n\}\ni z.$$

(1.8.9) **Proposition.** *Let $E_1, E_2, \cdots, E_m$ be a family of ranked vector spaces, $\{z_n\}=\{(x_{n1}, x_{n2}, \cdots, x_{nm})\}$, $\{z'_n\}=\{(x'_{n1}, x'_{n2}, \cdots, x'_{nm})\}$ two sequences in the direct product $\times E_i$ and $z=(x_1, x_2, \cdots, x_m)$, $z'=(x'_1, x'_2, \cdots, x'_m)\in\times E_i$. If $\{\mathrm{Lim}\, z_n\}\ni z$ and $\{\mathrm{Lim}\, z'_n\}\ni z'$, then*

$$\{\mathrm{Lim}\,(z_n+z'_n)\}\ni z+z'.$$

Proof. By (1.8.8) $\{\mathrm{Lim}\, z_n\}\ni z$ and $\{\mathrm{Lim}\, z'_n\}\ni z$ are equivalent to

$$\{\mathrm{Lim}\, x_{n1}\}\ni x_1, \{\mathrm{Lim}\, x_{n2}\}\ni x_2, \cdots, \{\mathrm{Lim}\, x_{nm}\}\ni x_m,$$

and

$$\{\mathrm{Lim}\, x'_{n1}\}\ni x'_1, \{\mathrm{Lim}\, x'_{n2}\}\ni x'_2, \cdots, \{\mathrm{Lim}\, x'_{nm}\}\ni x'_m.$$

Since $E_1, E_2, \cdots, E_m$ are ranked vector spaces, it follows from (1.8.4) that for each i $(i=1, 2, \cdots, m)$

$$\{\mathrm{Lim}\,(x_{ni}+x'_{ni})\}\ni x_i+x'_i$$

$$\therefore\quad \{\mathrm{Lim}\,(z_n+z'_n)\}\ni z+z'.$$

(1.8.10) **Proposition.** *Let $E_1, E_2, \cdots, E_m$ be a family of ranked vector spaces, $\{z_n\}=\{(x_{n1}, x_{n2}, \cdots, x_{nm})\}$ a sequence in the direct product $\times E_i$ and $z=(x_1, x_2, \cdots, x_m) \in \times E_i$. If $\{\text{Lim } z_n\} \ni z$ in $\times E_i$, then for any $\lambda \in \Re$*

$$\{\text{Lim } \lambda z_n\} \ni \lambda z.$$

Proof. By (1.8.8) $\{\text{Lim } z_n\} \ni z$ is equivalent to

$$\{\text{Lim } x_{n1}\} \ni x_1, \{\text{Lim } x_{n2}\} \ni x_2, \cdots, \{\text{Lim } x_{nm}\} \ni x_m.$$

Since $E_1, E_2, \cdots, E_m$ are ranked vector spaces, by (1.8.5), for any $\lambda \in \Re$, we have

$$\{\text{Lim } \lambda x_{n1}\} \ni \lambda x_1, \{\text{Lim } \lambda x_{n2}\} \ni \lambda x_2, \cdots, \{\text{Lim } \lambda x_{nm}\} \ni \lambda x_m.$$

$$\therefore \quad \{\text{Lim } \lambda z_n\} \ni \lambda z.$$

(1.8.11) **Proposition.** *Let $z=(x_1, x_2, \cdots, x_m)$ be an arbitrary element of the direct product $\times E_i$ of the ranked vector spaces $E_1, E_2, \cdots, E_m$. If $\lim \lambda_n = \lambda$ in $\Re$, then*

$$\{\text{Lim } \lambda_n z\} \ni \lambda z.$$

Proof. Since $E_1, E_2, \cdots, E_m$ are ranked vector spaces, by (1.8.6) we have

$$\{\text{Lim } \lambda_n x_1\} \ni \lambda x_1, \{\text{Lim } \lambda_n x_2\} \ni \lambda x_2, \cdots, \{\text{Lim } \lambda_n x_m\} \ni \lambda x_m.$$

$$\therefore \quad \{\text{Lim } \lambda_n z\} \ni \lambda z.$$

(1.8.12) **Proposition.** *Let $\{z_n\}=\{(x_{n1}, x_{n2}, \cdots, x_{nm})\}$ a sequence in the direct product $\times E_i$ of the ranked vector spaces $E_1, E_2, \cdots, E_m$ and $z=(x_1, x_2, \cdots, x_m) \in \times E_i$. If $\{\text{Lim } z_n\} \ni z$ in $\times E_i$ and $\lim \lambda_n = \lambda$ in $\Re$, then*

$$\{\text{Lim } \lambda_n z_n\} \ni \lambda z.$$

Proof. $\{\text{Lim } z_n\} \ni z$ is equivalent to

$$\{\text{Lim } x_{n1}\} \ni x_1, \{\text{Lim } x_{n2}\} \ni x_2, \cdots, \{\text{Lim } x_{nm}\} \ni x_m.$$

Since $E_1, E_2, \cdots, E_m$ are ranked vector spaces, by (1.8.7) we have

$$\{\text{Lim } \lambda_n x_{n1}\} \ni \lambda x_1, \{\text{Lim } \lambda_n x_{n2}\} \ni \lambda x_2, \cdots, \{\text{Lim } \lambda_n x_{nm}\} \ni \lambda x_m.$$

$$\therefore \quad \{\text{Lim } \lambda_n z_n\} \ni \lambda z.$$