69. On the Nörlund Summability of the Conjugate Series of Fourier Series

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§ 1. Let $\{p_n\}$ be a sequence such that $P_n=p_0+p_1+\cdots+p_n\neq 0$ for $n=0, 1, 2, \cdots$. A series $\sum_{n=0}^{\infty} a_n$ with its partial sum s_n is said to be summable (N, p_n) to sum s, if $(p_n s_0+p_{n-1}s_1+\cdots+p_0 s_n)/P_n \to s$ as $n\to\infty$. The choice $p_n=1/(n+1)$ yields the familiar harmonic summability. Let $f(t)$ be a periodic finite-valued function with period 2π and integrable (L) over $(-\pi, \pi)$. Let its Fourier series be

(1.1) $$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nt+b_n \sin nt)\equiv\sum_{n=0}^{\infty}A_n(t).$$

Then the conjugate series of the series (1.1) is

(1.2) $$\sum_{n=1}^{\infty}(b_n \cos nt-a_n \sin nt)\equiv\sum_{n=1}^{\infty}B_n(t).$$

Throughout this paper, we write

$$\varphi(t)\equiv\frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}, \qquad \Phi(t)\equiv\int_0^t|\varphi(u)|\,du,$$

$$\psi(t)\equiv\frac{1}{2}\{f(x+t)-f(x-t)\}, \qquad \Psi(t)\equiv\int_0^t|\psi(u)|\,du$$

and $\tau=[1/t]$, where $[\lambda]$ is the integral part of λ.

On the Nörlund summability of Fourier series at a given point x, the following results are known. Iyengar [3] proved that if

$$\varphi(t)=o(1/\log t^{-1}) \quad \text{as} \quad t\to+0,$$

then the series (1.1) at $t=x$ is harmonic summable to sum $f(x)$. Later, generalizing this result, Siddiqi [5] proved that if

$$\Phi(t)=o(t/\log t^{-1}) \quad \text{as} \quad t\to+0,$$

then the series (1.1) at $t=x$ is harmonic summable to sum $f(x)$. Further, generalizing this result, Pati [7] proved the following

Theorem A. *Let $\{p_n\}$ be a sequence such that*

$$p_n>0,\ p_n\downarrow,\ P_n\to\infty \quad \text{and} \quad \log n=O(P_n).$$

If

$$\Phi(t)=o(t/P_\tau) \quad \text{as} \quad t\to+0,$$

then the series (1.1) at $t=x$ is summable (N, p_n) to sum $f(x)$.

Furthermore Rajagopal [8] proved the following

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Theorem B. *Let a function $p(t)$ be monotone non-increasing and positive for $t \geqq 0$. Let $p_n = p(n)$ and let*

$$P(t) \equiv \int_0^t p(u)du \to \infty \quad as \quad t \to \infty. \tag{1.3}$$

If, for some fixed δ, $0 < \delta < 1$,

$$\int_{\frac{1}{n}}^{\delta} \Phi(t) \frac{d}{dt} \frac{P(1/t)}{t} dt = o(P_n) \quad as \quad n \to \infty, \tag{1.4}$$

then the series (1.1) *at $t = x$ is summable (N, p_n) to sum $f(x)$.*

It should be noted that Theorem B is a generalization of Theorem A. Thus, among these results, Theorem B is the most general. We now remark that, from Rajagopal [8, Lemma (a)], (1.3) and (1.4) together imply

$$\Phi(t) = o(t) \quad \text{as} \quad t \to +0. \tag{1.5}$$

On the other hand, the summability (N, p_n) of the conjugate series of Fourier series at a given point x has been considered by Siddiqi [6], Dikshit [1, 2], Saxena [4] and others, respectively. Siddiqi proved that if

$$\Psi(t) = o(t/\log t^{-1}) \quad \text{as} \quad t \to +0,$$

then the series (1.2) at $t = x$ is harmonic summable to sum

$$\tilde{f}(x) = -\frac{1}{\pi} \int_0^{\pi} \psi(t) \cot \frac{t}{2} \, dt \tag{1.6}$$

provided that the integral exists as a Cauchy integral at origin. A conjugate-analogue of Pati's Theorem A was obtained by Dikshit [1]. That result was generalized independently by Dikshit [2] and by Saxena [4]. Their theorems are as follows.

Theorem C. (Dikshit [2]). *Let $\{p_n\}$ be a sequence such that*

$$p_n > 0, \ p_n \downarrow, \ P_n \to \infty, \quad and \quad \alpha(n) \log n = O(P_n), \tag{1.7}$$

where $\alpha(t)$ is a positive monotone non-decreasing function. If

$$\Psi(t) = o(\alpha(1/t) t / P_\tau) \quad as \quad t \to +0, \tag{1.8}$$

then the series (1.2) *at $t = x$ is summable (N, p_n) to sum $\tilde{f}(x)$ provided that the integral in* (1.6) *exists as a Cauchy integral at origin.*

Theorem D. (Saxena [4]). *Let $\{p_n\}$ be a sequence such that*

$$p_n > 0, \ p_n \downarrow, \ P_n \to \infty, \quad and \quad \log n = O(\beta(P_n)), \tag{1.9}$$

where $\beta(t)$ is a positive monotone non-decreasing function such that $t/\beta(t)$ is also monotone non-decreasing. If

$$\Psi(t) = o(t/\beta(P_\tau)) \quad as \quad t \to +0, \tag{1.10}$$

then the series (1.2) *at $t = x$ is summable (N, p_n) to sum $\tilde{f}(x)$ provided that the integral in* (1.6) *exists as a Cauchy integral at origin.*

Remark. It is easily proved that the assumption on monotone of $\beta(t)$ is superfluous.

Theorem E. (Saxena [4]). *Let $\{p_n\}$ be a sequence such that*

$$p_n > 0, \ p_n \downarrow, \ P_n \to \infty, \quad and \quad \log n = O(\gamma(P_n)),$$

where $\gamma(t)$ is a positive function such that

$$\int_{\frac{1}{n}}^{\delta} \frac{P_\tau}{\gamma(P_\tau)} \frac{1}{t}\, dt = O(P_n) \quad as \quad n\to\infty.$$

If

$$\Psi(t) = o(t/\gamma(P_\tau)) \quad as \quad t\to +0,$$

then the series (1.2) at $t=x$ is summable (N, p_n) to sum $\tilde{f}(x)$ provided that the integral in (1.6) exists as a Cauchy integral at origin.

It is obvious that Theorem E is a generalization of Theorem D.

The purpose of this paper is to prove the following two theorems.

Theorem 1. *Let $\{p_n\}$ and $P(t)$ be defined as in Theorem B. If, for some fixed δ, $0<\delta<1$,*

$$\int_{\frac{1}{n}}^{\delta} \Psi(t) \frac{d}{dt} \frac{P(1/t)}{t}\, dt = o(P_n) \quad as \quad n\to\infty, \tag{1.11}$$

then the series (1.2) at $t=x$ is summable (N, p_n) to sum $\tilde{f}(x)$ provided that the integral in (1.6) exists as a Cauchy integral at origin.

Theorem 2. *If Theorem C holds, then Theorem D also holds and conversely, when*

$$\alpha(1/t)/\alpha(\tau) = O(1) \quad as \quad t\to +0, \tag{1.12}$$

if Theorem D holds, then Theorem C also holds.

Obviously there exists a function $\alpha(t)$ which does not satisfy the condition (1.12). Thus we see that Theorem C is better than Theorem D. We do not know however a relation between Theorems C and E, when the function $\alpha(t)$ does not satisfy the condition (1.12).

§2. Proof of Theorem 1. Let us write

$$\tilde{s}_n(x) = \sum_{k=1}^{n} B_k(x) \quad \text{and} \quad \tilde{t}_n(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \tilde{s}_k(x).$$

Then we have

$$\tilde{s}_n(x) - \tilde{f}(x) = \frac{1}{\pi} \int_0^{\pi} \psi(t) \frac{\cos\left(n+\frac{1}{2}\right)t}{\sin t/2}\, dt$$

$$= \frac{1}{\pi} \int_0^{\delta} \psi(t) \frac{\cos\left(n+\frac{1}{2}\right)t}{\sin t/2}\, dt + \eta_n,$$

where, by the Riemann-Lebesgue theorem,

$$\eta_n = \frac{1}{\pi} \int_{\delta}^{\pi} \psi(t) \frac{\cos\left(n+\frac{1}{2}\right)t}{\sin t/2}\, dt \to 0 \quad \text{as} \quad n\to\infty, \tag{2.1}$$

and

$$\tilde{t}_n(x) - \tilde{f}(x) = \sum_{k=0}^{n} p_{n-k}\{\tilde{s}_k(x) - \tilde{f}(x)\}/P_n$$

$$=\frac{1}{\pi P_n}\int_0^\delta \psi(t)\sum_{k=0}^n p_{n-k}\frac{\cos\left(k+\dfrac{1}{2}\right)t}{\sin t/2}dt+\xi_n$$
$$=\frac{1}{\pi P_n}\int_0^\delta \psi(t)\frac{K_n(t)}{\sin t/2}dt+\xi_n,$$

where $K_n(t)=\sum_{k=0}^n p_{n-k}\cos\left(k+\frac{1}{2}\right)t$ and, by (2.1) together with the regularity of the method of summation (N, p_n),

$$\xi_n=\frac{1}{P_n}\sum_{k=0}^n p_{n-k}\eta_k\to 0 \quad \text{as} \quad n\to\infty.$$

Thus we have

$$\tilde{t}_n(x)-\tilde{f}(x)=\frac{1}{\pi P_n}\left(\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^\delta\right)\psi(t)\frac{K(t)}{\sin t/2}dt+o(1)$$
$$=I_n+J_n+o(1),$$

say. Since the integral in (1.6) exists, we have

$$\frac{1}{\pi}\int_0^{\frac{1}{n}}\psi(t)\cos\frac{t}{2}dt=o(1) \quad \text{as} \quad n\to\infty.$$

Hence

$$I_n=\frac{1}{\pi P_n}\int_0^{\frac{1}{n}}\psi(t)\frac{K_n(t)}{\sin t/2}dt$$
$$=\frac{1}{\pi P_n}\int_0^{\frac{1}{n}}\psi(t)\frac{K_n(t)-P_n\cos t/2}{\sin t/2}dt+o(1)$$
$$=\frac{1}{\pi P_n}\int_0^{\frac{1}{n}}\psi(t)\left(\sum_{k=0}^n p_{n-k}\frac{\cos(k+1/2)t-\cos t/2}{\sin t/2}\right)dt+o(1).$$

Since $|\sin kt|\leqq k|\sin t|$ when k is an positive integer, we get

$$\sum_{k=0}^n p_{n-k}\frac{\cos(k+1/2)t-\cos t/2}{\sin t/2}=-2\sum_{k=0}^n p_{n-k}\frac{\sin(k+1)t/2\sin kt/2}{\sin t/2}$$
$$=O\Big(\sum_{k=0}^n kp_{n-k}(k+1)t\Big)$$
$$=O\Big(n^2t\sum_{k=0}^n p_{n-k}\Big)$$
$$=O(n^2P_nt).$$

Hence, by an analogue of (1.5),

$$I_n=O\left(\frac{1}{\pi P_n}\int_0^{\frac{1}{n}}|\psi(t)|n^2P_ntdt\right)+o(1)$$
$$=O\left(n\int_0^{\frac{1}{n}}|\psi(t)|dt\right)+o(1)=o(1) \quad \text{as} \quad n\to\infty.$$

Thus, in order to prove Theorem, it is sufficient to prove that $J_n=o(1)$ as $n\to\infty$. But this is proved by an estimation similar to one of J_n in the proof of Theorem B. Thus our Theorem is completely proved.

§ 3. **Proof of Theorem 2.** *Let $h(t)$ be a continuous function defined over $t \geqq 0$ such that*

$$h(t)=P_n \quad (t=n) \quad \text{and} \quad h(t)=\text{linear} \quad (\text{elsewhere}).$$

Then the function $h(t)$ is strictly increasing so that the function $h(t)$ has a inverse function $k(t)$ such that $n=k(P_n)$ and $k(t)$ is strictly increasing. We shall now prove the first part of Theorem. Let $\{p_n\}$ and $\beta(t)$ satisfy the condition of Theorem D. Then we define $\alpha(t)$ by $\alpha(t)=h(t)/\beta(h(t))$. Since the functions $h(t)$ and $t/\beta(t)$ are monotone non-decreasing, the function $\alpha(t)$ is also so. Then, by (1.9),

$$\log n = O(\beta(P_n)) = O(P_n/\alpha(n))$$

and, by (1.10),

$$\Psi(t)=o(t/\beta(P_\tau))=o(\alpha(\tau)t/P_\tau)=o(\alpha(1/t)t/P_\tau).$$

These prove the first part of Theorem. To prove the converse part, we set $\beta(t)=t/\alpha(k(t))$. Then we see that $t/\beta(t)=\alpha(k(t))$ is monotone non-decreasing and, by (1.7),

$$\log n = O(P_n/\alpha(n)) = O(P_n/\alpha(k(P_n))) = O(\beta(P_n)),$$

and, by (1.8) and (1.12),

$$\Psi(t)=o(\alpha(1/t)t/P_\tau)=o(\alpha(\tau)t/P_\tau)=o(\alpha(k(P_\tau))t/P_\tau)=o(t/\beta(P_\tau)).$$

Thus the proof is complete.

§ 4. We shall now show that *Theorem 1 is a generalization of Theorem C.* For the proof, it is sufficient to prove that the condition of Theorem C implies the one of Theorem 1. Let $\{p_n\}$ be given as in Theorem C. Then we define a function $p(t)$ by

$$p(t)=p_n \quad \text{for} \quad n \leqq t < n+1, \qquad n=0, 1, 2, \cdots.$$

Further define a function $P(t)$ as in (1.3). Then, by the condition, the function $p(t)$ is monotone non-decreasing and positive for $t \geqq 0$. By (1.7) and (1.8), we get, $t \to +0$,

$$P(1/t) \to \infty, \quad P_\tau \sim P(1/t), \quad \alpha(1/t)/P(1/t)=O(1/\log t^{-1}),$$

and

$$\Psi(t)=o(\alpha(1/t)t/P(1/t)).$$

Hence we have

$$\begin{aligned}\int_{\frac{1}{n}}^{\delta} \Psi(t)\frac{d}{dt}\frac{P(1/t)}{t}dt &= o\left(\int_{\frac{1}{n}}^{\delta} \alpha\left(\frac{1}{t}\right)\frac{t}{P(1/t)}\frac{d}{dt}\frac{P(1/t)}{t}dt\right) \\ &= o\left(\int_{\frac{1}{n}}^{\delta} \alpha(t)\frac{p(1/t)}{P(1/t)}\frac{1}{t^2}dt + \int_{\frac{1}{n}}^{\delta} \alpha\left(\frac{1}{t}\right)\frac{1}{t}dt\right) \\ &= o\left(\int_{\frac{1}{\delta}}^{n} \frac{p(t)}{\log t}dt\right) + o(P_n) \\ &= o(P_n)\end{aligned}$$

which shows that (1.11) holds. Thus the proof is complete.

We shall next show that *Theorem 1 is also a generalization of Theorem E.* Let $\{p_n\}$ be given as in Theorem E. Then we define functions $p(t)$ and $P(t)$ as in the above case. Since the sequence $\{p_n\}$ is monotone decreasing,

$$\frac{1}{t}p\left(\frac{1}{t}\right)<(n+1)p_n\leqq P_n=P_\tau\leqq P\left(\frac{1}{t}\right),$$

when $n\leqq 1/t<n+1$, $n=0, 1, 2, \cdots$. Thus we have, by the condition,

$$\int_{\frac{1}{n}}^{\delta}\Psi(t)\frac{d}{dt}\frac{P(1/t)}{t}dt=o\left(\int_{\frac{1}{n}}^{\delta}\frac{p(1/t)}{\gamma(P_\tau)}\frac{1}{t^2}dt+\int_{\frac{1}{n}}^{\delta}\frac{P(1/t)}{\gamma(P_\tau)}\frac{1}{t}dt\right)$$
$$=o\left(\int_{\frac{1}{n}}^{\delta}\frac{P(1/t)}{\gamma(P_\tau)}\frac{1}{t}dt\right)$$
$$=o(P_n),$$

which shows that (1.11) holds. Thus the proof is complete.

§ 5. From the argument in § 4, we have the following theorems as corollaries of Theorem B. These are analogues of Theorems C and E.

Theorem 3. *Let* $\{p_n\}$ *and* $\alpha(t)$ *be defined as in Theorem C. If*

$$\Phi(t)=o(\alpha(1/t)t/P) \quad as \quad t\rightarrow+0,$$

then the series (1.1) *at* $t=x$ *is summable* (N, p_n) *to sum* $f(x)$.

Theorem 4. *Let* $\{p_n\}$ *and* $\gamma(t)$ *be defined as in Theorem E. If*

$$\Phi(t)=o(t/\gamma(P_\tau)) \quad as \quad t\rightarrow+0,$$

then the series (1.1) *at* $t=x$ *is summable* (N, p_n) *to sum* $f(x)$.

It should be noted that these Theorems are also proved directly by analogous methods to those of the proofs of Theorems C and E.

References

[1] H. P. Dikshit: The Nörlund summability of the conjugate series of a Fourier series. Rend. Circ. Mat. Palermo, **11**, 217-224 (1962).
[2] ——: On the Nörlund summability of the conjugate series of a Fourier series. Rend. Circ. Mat. Palermo, **14**, 165-170 (1965).
[3] K. S. K. Iyengar: A Tauberian theorem and its application to convergence of Fourier series. Proc. Indian Acad. Sci., Sec. A, **18**, 81-87 (1943).
[4] A. Saxena: On Nörlund summability of the conjugate series of a Fourier series. Rend. Circ. Mat. Palermo, **13**, 222-228 (1964).
[5] J. A. Siddiqi: On the harmonic summability of Fourier series. Proc. Indian Acad. Sci., Sec. A, **28**, 527-531 (1948).
[6] ——: The Nörlund summability of trigonometrical series. D. Phil. Thesis. (1949).
[7] T. Pati: A generalization of a theorem of Iyengar on the harmonic summability of Fourier series. Indian J. Math., **3**, 85-90 (1961).
[8] C. T. Rajagopal: On the Nörlund summability of Fourier series. Proc. Cambridge Phil. Soc., **59**, 47-53 (1963).