109. A Characterization of Best Tchebycheff Approximations in Function Spaces

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1. Introduction. The purpose of this paper is to prove a theorem which characterizes best Tchebycheff approximations in linear subspaces of the space of all continuous real- or complex-valued functions defined on a compact topological space. There is a recent theorem due to I. Singer [1, Theorem 5] which treats the identical problem. The theorem presented in this paper differs from his in that we characterize best approximations in terms of the evaluation functionals corresponding to those points (the critical points) at which the error function (i.e., approximant-approximator) attains its maximum deviation from zero. Those evaluation functionals are, incidentally, precisely the extreme points of the unit sphere of the conjugate space [3, Problem J, p. 134]. The present theorem is also a direct generalization of a known theorem which treats the case where the approximating functions form a finite-dimensional subspace (see, for example, [4]). We shall state this special case as a corollary to the main theorem.

Actually, the main theorem on the part of necessity was first proved for reflexive subspaces. It is Dr. E. W. Cheney who kindly showed in his letter to the author that the original proof could be improved with a slight modification to include all linear subspaces. The author would like to express his gratitude to Dr. Cheney for this and other valuable suggestions.

Let X be a given compact topological space; $C(X)$, the space of all real- or complex-valued continuous functions defined on X with the Tchebycheff norm:

$$||f|| = \max\{|f(x)| : x \in X\}.$$

Let M be a proper subspace of $C(X)$; x^*, the evaluation functional corresponding to an element x in X, that is,

$$x^*(f) = f(x) \text{ for all } f \text{ in } C(X).$$

2. Main theorem. Let f be an element of $C(X)\backslash M$ and p an element of M. Let r denote the error f-p. Denote by K the set of

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all critical points of r:

$$K=\{x \in X : |r(x)|=||r||\}.$$

Let

$$Q=\{\overline{r(x)}(x^* | M) : x \in K\}$$

where $x^* | M$ denotes the restriction of x^* to M. Then, in order that the element p be a best approximation in M to f, it is necessary and sufficient that the origin of M^* shall lie in the $w(M^*, M)$-closure of the convex hull of Q.

Proof. *Sufficiency*. Suppose that the stated condition is fulfilled. We shall show the existence of an element F of $C^*(X)$ having the following properties:

(a) $||F||=1$

(b) F annihilates M

(c) $|F(f)|=||r||$.

This will show that p is a best approximation in M to f (actually, Conditions (a)-(c) are both necessary and sufficient in order that p be a best approximation in M to f[2, p. 182]). In fact, assuming Conditions (a)-(c) we have for each element q in M

$$||f-q||=||F|| \; ||f-q|| \geq |F(f-q)|=|F(f)|=||r||=||f-p||.$$

Now, from the hypothesis, the origin of M^* is the $w(M^*, M)$-limit of some net, say F_α, in the convex hull of Q. Each F_α is a convex combination of a finite number of elements in Q:

$$F_\alpha=\sum d_i \overline{r(x_i)}(x^* | M)$$

where $d_i \geq 0$ and $\sum d_i=1$. Consider the new net G_α in $C^*(X)$ defined by

$$G_\alpha=\sum d_i \overline{r(x_i)} x_i^* .$$

(G_α is an extension of F_α.) The net G_α is norm-bounded in $C^*(X)$ (in fact, by $||r||$). Then, by the Alaoglu theorem [5, p. 424], the net G_α has a subnet, say G_{α_β}, which converges in the ordinary w^* topology on $C^*(X)$, to an element, say G, of $C^*(X)$ such that $||G|| \leq ||r||$. The functional G annihilates M; in fact, for each q in M, we have

$$G(q)=\lim G_{\alpha_\beta}(q)=\lim F_{\alpha_\beta}(q)=0.$$

We also have $G(r)=||r||^2$, for

$$G_\alpha(r)=\sum d_i \overline{r(x_i)} x_i^*(r)=\sum d_i ||r||^2=||r||^2$$

and

$$G(r)=\lim G_{\alpha_\beta}(r)=\lim ||r||^2=||r||^2.$$

We already know that $||G|| \leq ||r||$. Hence the above equality shows that $||G||=||r||$. By setting $\mathrm{F}=\mathrm{G}/||r||$ (where $||r||>0$ since $f \neq p$), we easily see that F satisfies all Properties (a)-(c) mentioned at the beginning of this proof.

Necessity. Suppose that the proposed condition fails. Then by [3, 14.4, p. 118], there exists an $w(M^*, M)$-continuous linear functional, say H, whose real part strongly separates the origin of M^* from the $w(M^*, M)$-closure of the convex hull of Q. But, by [3, 16.2, p. 140], such functional H is represented by some element in M. Consequently, there exists an element h in M such that

$$\inf \{\mathrm{Re}[\overline{r(x)}h(x)] : x \in K\} \equiv d > 0.$$

We will show that for a sufficiently small positive real number λ, the function $p+\lambda h$ is a strictly better approximation in M to f than p is, showing that the proposed condition is also necessary. Thus, let

$$A = \{x \in X : \mathrm{Re}[\overline{r(x)}h(x)] > d/2\}.$$

Then A is open and contains K. The complement $X\backslash A$ is therefore compact and is disjoint from K. Hence for some $d' > 0$

$$\sup \{|r(x)| : x \in X/A\} = ||r|| - d'.$$

Choose λ such that $0 < \lambda < \min \{2d/||h||^2, d'/||h||\}$. Then on A,

$$\begin{aligned} |r-\lambda h|^2 &= |r|^2 - 2\lambda \mathrm{Re}[\bar{r}h] + \lambda^2 |h|^2 \\ &< |r|^2 + \lambda\{-2d + (2d/||h||^2)||h||^2\} \\ &\leq ||r||^2 \end{aligned}$$

and, on $X\backslash A$,

$$|r-\lambda h| \leq |r| + \lambda |h| < ||r|| - d' + (d'/||h||)\ ||h|| = ||r||.$$

Consequently, $|r-\lambda h| = |f-(p+\lambda h)| < ||r||$ on the entire set X, whence $p+\lambda h$ is a strictly better approximation in M to f than p is.

3. **Corollary.** *If M is finite-dimensional, then p is a best approximation in M to f if and only if 0 lies in the convex hull of the set (in the n-space)*

$$\{\overline{r(x)}[g_1(x), \cdots g_n(x)] : |r(x)| = ||r||\}$$

where $r = f - p$ (as before) and $\{g_1, \cdots, g_n\}$ is some basis for M.

Proof. The set Q in the main theorem is an $w(M^*, M)$-compact subset of M^* because the set K is a compact subset of X. Since M is finite-dimensional, so is M^*. By [6, Lemma 1, p. 39], [6, Theorem 1, p. 40] and [7, Theorem 10, p. 22], the convex hull of Q is also $w(M^*, M)$-compact, hence $w(M^*, M)$-closed. Then, the condition proposed in the main theorem is easily seen to be equivalent to the one given in the present corollary.

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