98. *Generalizations of the Alaoglu Theorem with Applications to Approximation Theory. I*[*)]

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1. Introduction. This paper concerns the approximation of set-valued maps by multilinear maps on direct sums of normed linear spaces. In particular, we prove two very general existence theorems for best approximations in the sense of Tchebycheff (Theorems 3 and 7). One of these theorems (Theorem 7) generalizes one in [1, p. 97] and can be interpreted as a generalization also of the Alaoglu theorem [2, p. 424]. We recall that a real-valued function defined on a topological space X is called *lower-semicontinuous* if each set of the form $\{x \in X : f(x) \leq c\}$ is closed. The fact that a lower-semicontinuous function defined on a compact topological space is bounded below and achieves its infinum thereon will be needed frequently, and will be used freely without explicit reference.

2. Definitions. Let E and F be normed linear spaces over the same scalar field. Let Q be a set-valued map of $X (X \subseteq E)$ into F. We use the symbol $||Q||_X$ to denote the number $\sup\{||p|| : p \in Qx$ for some $x \in X\}$. We say that Q is *bounded* if $||Q||_X$ is finite. Let K be another set-valued map of X into F. We define $K-Q$ in the most natural way, that is $(K-Q)(x)=Kx-Qx=\{p-q : p \in Kx$ and $q \in Qx\}$. Let M be a family of set-valued maps of X into F. A member P_0 of M is termed a *best approximation in M to Q* if $||P_0-Q||_X=\inf\{||P-Q||_X : P \in M\}$.

Let $E_1, \cdots, E_n, F$ be normed linear spaces over the same scalar field. The direct sum of the spaces $E_1, \cdots, E_n$ is the normed linear space of all ordered n-tuples $[x_1, \cdots, x_n]$, where $x_i \in E_i$, $i=1, \cdots, n$, with component wise addition and component wise scalar multication, and with the norm defined by $||[x_1, \cdots, x_n]||=\max\{||x_i||\ E_i : i=1, \cdots, n\}$. We denote this direct sum by the symbol $E_1 \oplus \cdots \oplus E_n = E$.

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Let $u(E, F)$ denote the set of all n-linear maps of the direct sum E into the normed linear space F, and let $b(E, F)$ denote the subset of $u(E, F)$ consisting of all continuous maps in $u(E, F)$. The space $u(E, F)$ is a linear space and the space $b(E, F)$ is a normed linear space with the norm defined by $||A|| = \sup\{||Ax|| : ||x|| \leq 1, x \in E\}$. Let E_1 and E_2 be normed linear spaces. We denote by $B(E_1, E_2)$ the space of all bounded linear maps of E_1 into E_2.

Let $E_1, \cdots, E_n, F$ be normed linear spaces and let $E = E_1 \oplus \cdots \oplus E_n$. The *$\sigma$-topology* on $b(E, F)$ is the topology with respect to which a net A_α in $b(E, F)$ converges to $A \in b(E, F)$ if and only if the net $A_\alpha x$ converges to Ax(in F) for all $x \in E$. Let $X \subseteq E$. The *σ_X-topology* on $u(E, F)$ is the topology with respect to which a net A_α in $u(E, F)$ converges to $A \in u(E, F)$ if and only if the net $A_\alpha x$ converges to Ax (in F) for all $x \in X$. The *τ-topology* on $B(E_1, E_2^*)$ is the topology with respect to which a net A_α in $B(E_1, E_2^*)$ converges to A if and only if $\langle y, A_\alpha x \rangle \to \langle y, Ax \rangle$ for all $x \in E_1$ and all $y \in E_2$, where $\langle , \rangle$ denotes the ordinary dual pairing.

3. **Theorem.** *Let $E_1, \cdots, E_n$ be normed linear spaces and let F be a finite-dimensional normed linear space. Let E denote the direct sum $E_1 \oplus \cdots \oplus E_n$. Let X be an arbitrary non-empty subset of E. Let Q be a set-valued bounded map of E into F. Then, for each $k \geq 0$, the set $M_k = \{A \in u(E, F) : ||A - Q||_X \leq k\}$ is σ_X-compact and σ_X-closed.*

Proof. In order to prove that M_k is σ_X-compact, let A_α be a net in M_k. It is sufficient to find a subnet A_{α_β} (of the net A_α) which converges in the σ_X-topology to an element of M_k. Thus for all $A \in M_k$ and all $x \in X$, $||Ax|| \leq ||Q||_X + k = k'$. Let

$$R = \{f \in F : ||f|| \leq k'\} \subseteq F.$$

Then, for each α, $$A_\alpha | X \in R^X$$
where the notation $A_\alpha | X$ means the restriction of A_α to X. The set R is compact in the space F (it is here that we use the fact that the space F is finite-dimensional). By the Tychonoff theorem the product space R^X is compact in the product topology. Therefore, the net $A_\alpha | X$ has a subnet $A_{\alpha_\beta} | X$ which converges to an element A' of the product space in the product topology, that is, pointwise on X. We will first show that A' is n-linear on the set X. Let λ and λ' be scalars and let $[x_1, x_2, \cdots, x_n] \in X$, $[x_1', x_2, \cdots, x_n] \in X$, and $[\lambda x_1 + \lambda' x_1', x_2, \cdots, x_n] \in X$. Then $A'[\lambda x_1 + \lambda' x_1', x_2, \cdots, x_n] = \lim A_{\alpha_\beta}[\lambda x_1 + \lambda' x_1', x_2, \cdots, x_n] = \lim \lambda(A_{\alpha_\beta} [x_1, \cdots, x_n]) + \lim \lambda A_{\alpha_\beta}[x_1', \cdots, x_n] = \lambda A'[x_1, \cdots, x_n] + \lambda' A'[x_1', \cdots, x_n]$. This proves that A' is linear in the first component. Similarly, we can prove that A' is linear in any other component. We will next show that it is possible to extend A', without disturbing the n-line-

arity, to the entire space E. Thus, let $P_i(i=1, \cdots, n)$ be the projection which assigns to each $x \in E$ its i-th component. Let $H_i(i=1, \cdots \cdots, n)$ be a maxial linearly independent subset of the set P_iX, and let $H_i \cup K_i \equiv L_i$ be a maxial linearly independent subset of E_i. Then, it is easy to see that the following rule gives an n-linear map A'' of E into F which agrees with A' on X: $A''=A'$ on X and specify arbitrarily the value of A'' at each point in the union $\cup\{L_1x \cdots xL_{i-1}xK_ixL_{i+1}x \cdots \cdots xL_n : i=1, \cdots, n\}$. Since, for each $x \in X$ and each $p \in Qx$,

$$||A''x-p||=||A'x-p||=\lim_{\beta}||A_{\alpha_\beta}x-p||\leq k$$

we have $A'' \in M_k$. Also, the convergence in the product topology and that in the σ_X-topology agree on the set M_k. Hence $A_{\alpha_\beta} > A''$ in the σ_X-topology.

Using nets, the fact that M_k is σ_X-closed is easily proved.

4. **Corollary.** *Let E, F, X, and Q have the same meaning as in the last theorem. Then the map Q has a best approximation in $u(E, F)$, or more generally, in any σ_X-closed subset of $u(E, F)$.*

5. **Corollary.** *Let E be a normed linear space and let X be a nonempty subset of it. Let E' be the algebraic dual of E, that is, the space of all linear functionals on E. Then, the set $\{f' \in E' : |f'(x)| \leq$ for all $x \in X\}$ is σ_X-compact and σ_X-closed, where the σ_X-topology is the topology on E' with respect to which a net f'_α in E' converges to $f' \in E'$ if and only if $f'_\alpha(x) \to f'(x)$ for all $x \in X$.*

Proof. Take, in the last theorem, $n=1$, $F=$the scalar field and $Q=0$.

6. **Corollary (Alaoglu theorem).** *The unit sphere of the dual space of a normed linear space is w^*-compact.*

Proof. Take, in the last corollary, X to be the unit sphere of the given normed linear space.